

MODULI SPACES OF NONCOMMUTATIVE INSTANTONS: GAUGING AWAY NONCOMMUTATIVE PARAMETERS

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ABSTRACT. Using the theory of noncommutative geometry in a braided monoidal category, we improve upon a previous construction of noncommutative families of instantons of arbitrary charge on the deformed sphere S_θ^4 . We formulate a notion of noncommutative parameter spaces for families of instantons and we explore what it means for such families to be gauge equivalent, as well as showing how to remove gauge parameters using a noncommutative quotient construction. Although the parameter spaces are *a priori* noncommutative, we show that one may always recover a classical parameter space by making an appropriate choice of gauge transformation.

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1. INTRODUCTION

We study families of instantons on the noncommutative four-sphere S_θ^4 of [7] and what it means for such families to be gauge equivalent. We show that, although it is perfectly natural to allow for the possibility of families of instantons parameterised by noncommutative spaces, these ‘noncommutative parameters’ may always be removed by an appropriate choice of gauge transformation so as to recover a ‘commutative’ parameter space.

The study of instantons on S_θ^4 was initiated in [9, 10] and developed further in [11, 3], where it was observed that one may construct families of instantons which are parameterised by noncommutative spaces. These parameter spaces arise in a very natural way and suggest that we should consider seriously the idea that the moduli space of instantons might also be noncommutative. On the other hand, instantons on S_θ^4 are defined in terms of absolute minima of the Yang-Mills energy functional; thinking of the moduli space as being ‘the set of all such minima modulo gauge equivalence’ naturally leads us to expect it to be a classical space. Our goal in the present article is to use gauge theory to reconcile this apparent dichotomy between classical and noncommutative parameter spaces.

The driving force behind our investigation is the fact that the quantum sphere S_θ^4 can be obtained from its classical counterpart S^4 by means of a Hopf cocycle twisting procedure. The often-forgotten feature of this construction is that it deforms not just the four-sphere but in fact the entire category in which it lives. In our case, with $H = \mathcal{A}(\mathbb{T}^2)$ the Hopf algebra of coordinate functions on the two-torus \mathbb{T}^2 , the deformation takes the form of a ‘quantisation functor’ from the category of H -comodules, wherein lives the classical sphere S^4 , to a new category containing the quantum sphere S_θ^4 . This new category is the category of comodules for a twisted Hopf algebra H_F , with F a twisting Hopf cocycle.

By expressing the construction of instantons on S^4 entirely in this categorical framework, we are able to apply the quantisation functor and hence obtain a construction of instantons on S_θ^4 . Since all parameter spaces we consider are themselves objects in the category, they are twisted as well by the functor and we are naturally led to the concept of noncommutative families of instantons. We discuss what it means for such families to be gauge equivalent and show, just as in the classical case, how one can quotient parameter spaces by the resulting equivalence relation. As mentioned, a suitable choice of gauge transformation can be used to remove the noncommutativity of the parameters and produce an equivalent description in terms of usual spaces.

The paper is organised as follows. After section §2, which reviews the abstract theory of Hopf algebras and the cocycle twisting construction, we give an overview of how to construct the various noncommutative spaces that we shall need. In particular, §3 recalls

the construction of the noncommutative $SU(2)$ Hopf fibration $S_\theta^7 \rightarrow S_\theta^4$ using cocycle twisting, together with the canonical differential structures on these spaces.

As a way to understand the structures involved, the first topic of the paper will be to study parameter spaces for charge one instantons. In the classical case, one can construct all such instantons by acting upon a basic instanton with the group $SL(2, \mathbb{H})$ of conformal transformations of the four-sphere S^4 . In §4 and §5 we write the various symmetry groups of S^4 in an entirely H -covariant setting, which we then twist using the quantisation functor. This leads naturally to ‘braided geometry’ in the deformed category and, in particular, to a braided Hopf algebra $\mathcal{B}(SL_\theta(2, \mathbb{H}))$ of conformal symmetries. It obeys the usual axioms of a Hopf algebra, but with its structure maps required to be morphisms in the category. To this we apply a cobosonisation process to recover an ‘ordinary’ Hopf algebra, which takes the form of a Hopf algebra biproduct $\mathcal{B}(SL_\theta(2, \mathbb{H})) \bowtie H_F$.

In §6 we review the basic notions of gauge theory on S_θ^4 , then generalise them by formulating a notion of noncommutative parameter spaces for families of instantons and what it means for such families to be gauge equivalent. Using these definitions, we are able to parallel the classical case by interpreting the quantum symmetry group $\mathcal{B}(SL_\theta(2, \mathbb{H})) \bowtie H_F$ as a parameter space for the set of charge one instantons on the quantum four-sphere. In §7 we study this noncommutative parameter space in more detail, seeking where possible to remove all parameters corresponding to gauge equivalence instantons. In the classical case, the gauge parameters are described by the subgroup $Sp(2)$ of $SL(2, \mathbb{H})$ consisting of isometries of the sphere, so the ‘true’ parameter space is the quotient $SL(2, \mathbb{H})/Sp(2)$. In the noncommutative case there is a braided group of isometries $\mathcal{B}(Sp_\theta(2))$ that we are immediately able to remove by means of a quantum quotient construction.

Far more subtle is the question of how to remove the gauge parameters corresponding to the subalgebra H_F of $\mathcal{B}(SL_\theta(2, \mathbb{H})) \bowtie H_F$. These extra symmetries correspond to the inner automorphisms of the coordinate algebra $\mathcal{A}(S_\theta^4)$ of the deformed 4-sphere and constitute a very important part of its noncommutative geometry [6]. We show that there are many ways in which to quotient H_F away from $\mathcal{B}(SL_\theta(2, \mathbb{H})) \bowtie H_F$; in the classical case, every way we do this gives the same answer, but in the noncommutative case we get families of parameter spaces which are clearly different (some being quantum, some classical) but all have the same classical limit. We show that these parameter spaces are all gauge equivalent, finding as special cases both a commutative parameter space as well as the noncommutative parameter space found previously in [11].

In §8 we see how this method generalises to instantons with higher charge. We review the usual ADHM construction of [2] (*cf.* also [1]) in the context of braided geometry, which we then deform using the quantisation functor. With a few minor differences this essentially reproduces the noncommutative ADHM construction of [3], although derived from a different and arguably more natural approach. As in the charge one case, we show how to remove the gauge parameters corresponding to the torus algebra H_F , finding in particular that a certain choice of gauge yields again a commutative parameter space.

The paper concludes with an appendix reviewing the notion of quantum families of maps, which is an essential theme used throughout the paper. In looking for moduli spaces of instantons, our philosophy is to look not for a set of objects but rather for a space which parameterises those objects, that is to say we ask for some geometric

structure. In categorical terms, this means defining a functor from the category of algebras to the category of sets and then looking for the moduli space as a universal object; this is necessarily an object in the source category, *i.e.* an algebra.

2. PRELIMINARIES ON HOPF ALGEBRAS AND THEIR DEFORMATIONS

We review here some important elements of Hopf algebra theory, including the cocycle twisting construction that will play such an important part in what follows.

2.1. Hopf algebra preliminaries. We recall some basic facts from the theory of Hopf algebras and related structures following mainly [15] (*cf.* also [13]). Given a Hopf algebra H over \mathbb{C} we denote its coproduct, counit and antipode by $\Delta : H \rightarrow H \otimes H$, $\epsilon : H \rightarrow \mathbb{C}$ and $S : H \rightarrow H$, respectively. The product map is usually suppressed, although when explicitly written it is denoted $m(g \otimes h) = gh$. We use Sweedler notation for the coproduct, $\Delta h = h_{(1)} \otimes h_{(2)}$; also we indicate $(\Delta \otimes \text{id}) \circ \Delta h = (\text{id} \otimes \Delta) \circ \Delta h = h_{(1)} \otimes h_{(2)} \otimes h_{(3)}$ and so on, with summation inferred. A Hopf algebra H is said to be *coquasitriangular* if it is equipped with a convolution-invertible Hopf bicharacter $\mathcal{R} : H \otimes H \rightarrow \mathbb{C}$ satisfying

$$g_{(1)} h_{(1)} \mathcal{R}(h_{(2)}, g_{(2)}) = \mathcal{R}(h_{(1)}, g_{(1)}) h_{(2)} g_{(2)}$$

for all $g, h \in H$. Convolution invertibility is the existence of a map $\mathcal{R}^{-1} : H \otimes H \rightarrow \mathbb{C}$ such that

$$(2.1) \quad \mathcal{R}(h_{(1)}, g_{(1)}) \mathcal{R}^{-1}(h_{(2)}, g_{(2)}) = \mathcal{R}^{-1}(h_{(1)}, g_{(1)}) \mathcal{R}(h_{(2)}, g_{(2)}) = \epsilon(g) \epsilon(h)$$

for all $g, h \in H$. On the other hand being a bicharacter means that

$$(2.2) \quad \mathcal{R}(fg, h) = \mathcal{R}(f, h_{(1)}) \mathcal{R}(g, h_{(2)}), \quad \mathcal{R}(f, gh) = \mathcal{R}(f_{(1)}, h) \mathcal{R}(f_{(2)}, g)$$

for all $f, g, h \in H$. If in addition \mathcal{R} obeys the identity

$$\mathcal{R}(b_{(1)}, a_{(1)}) \mathcal{R}(a_{(2)}, b_{(2)}) = \epsilon(a) \epsilon(b)$$

for all $a, b \in H$ then we say that H is a *cotriangular* Hopf algebra.

A left module structure $H \otimes A \rightarrow A$ on a vector space A is denoted \triangleright , *i.e.* we write $h \otimes a \mapsto h \triangleright a$ for $h \in H$, $a \in A$. A right module structure is denoted \triangleleft . Similarly we denote a left comodule structure on A by $\Delta_L : A \rightarrow H \otimes A$, again using Sweedler notation $\Delta_L(a) = a^{(-1)} \otimes a^{(0)}$. A right comodule structure is written $\Delta_R : A \rightarrow A \otimes H$ with a similar Sweedler notation: $\Delta_R(a) = a^{(0)} \otimes a^{(+1)}$. We denote the categories of left H -modules and left H -comodules by ${}_H\mathcal{M}$ and ${}^H\mathcal{M}$ respectively. Moreover, we say A is a *left crossed H -module* if it is both a left H -module and a left H -comodule and these structures obey the compatibility condition

$$(2.3) \quad h_{(1)} a^{(-1)} \otimes h_{(2)} \triangleright a^{(0)} = (h_{(1)} \triangleright a)^{(-1)} h_{(2)} \otimes (h_{(1)} \triangleright a)^{(0)}$$

for all $h \in H$ and $a \in A$. The category of left crossed H -modules is denoted ${}^H_H\mathcal{C}$, with a similar definition for the category \mathcal{C}_H^H of right crossed H -modules. When H is coquasitriangular, there is a canonical monoidal functor ${}^H\mathcal{M} \rightarrow {}^H_H\mathcal{C}$ given by equipping an H -comodule A with the H -action

$$(2.4) \quad h \triangleright a := \mathcal{R}(a^{(-1)}, h) a^{(0)}, \quad a \in A, \quad h \in H,$$

where $a \mapsto a^{(-1)} \otimes a^{(0)}$ denotes the H -coaction, as before. One may check that this gives a well-defined H -action using the fact that \mathcal{R} is a Hopf bicharacter.

A monoidal (or tensor) category is *braided* if for each pair of objects V, W there is an isomorphism $\Psi_{V,W} : V \otimes W \rightarrow W \otimes V$, obeying certain natural hexagon identities [8]. The simplest example is the category \mathbf{Vec} of complex vector spaces, with the monoidal structure given by the usual tensor product of vector spaces and braided by the flip map: $\Phi_{V,W}(v \otimes w) = w \otimes v$ for all $v \in V$ and $w \in W$. More generally, if H is a Hopf algebra then the category ${}^H\mathcal{M}$ has a monoidal structure given by the tensor product coaction,

$$(2.5) \quad \Delta_{V \otimes W}(v \otimes w) = v^{(-1)}w^{(-1)} \otimes v^{(0)} \otimes w^{(0)}, \quad v \in V, w \in W.$$

If in addition H is coquasitriangular then ${}^H\mathcal{M}$ is braided by the collection of morphisms

$$(2.6) \quad \Psi_{V,W}(v \otimes w) = \mathcal{R}(w^{(-1)}, v^{(-1)})w^{(0)} \otimes v^{(0)},$$

for each pair V, W of left H -comodules with $v \in V$ and $w \in W$. In particular, if A and B are left H -comodule algebras (*i.e.* algebras in the category ${}^H\mathcal{M}$), the braiding allows one to give a tensor product algebra $A \underline{\otimes} B$, with the product

$$(a \otimes b)(c \otimes d) = a\Psi_{B,A}(b \otimes c)d,$$

and which lives in the category ${}^H\mathcal{M}$ by the coaction $\Delta_{A \underline{\otimes} B}$ in (2.5) above. The symbol $\underline{\otimes}$ is to remind us that the tensor product is the braided one. The braided monoidal category $(\mathbf{Vec}, \otimes, \Phi)$ is recovered by putting $H = \mathbb{C}[\mathbb{C}]$, the coordinate algebra of the complex numbers, with its trivial coquasitriangular structure.

If A is a left H -module algebra (*i.e.* an algebra in the category ${}_H\mathcal{M}$), then there is a cross product algebra $A \rtimes H$ built on $A \otimes H$ as a vector space, with algebra structure

$$(2.7) \quad (a \otimes g)(b \otimes h) := a(g_{(1)} \triangleright b) \otimes g_{(2)}h$$

for all $g, h \in H$ and $a, b \in A$; and unit $1_H \otimes 1_A$. Equally well, if A is a left A -comodule coalgebra (*i.e.* a coalgebra in the category ${}^H\mathcal{M}$), there is a cross coproduct coalgebra $A \rtimes H$ built on $A \otimes H$ as a vector space, with counit $\epsilon = \epsilon_A \otimes \epsilon_H$ and coproduct

$$(2.8) \quad \Delta(a \otimes h) := a_{(1)} \otimes a_{(2)}^{(-1)}h_{(1)} \otimes a_{(2)}^{(0)} \otimes h_{(2)}$$

for all $h \in H$ and $a \in A$.

Furthermore, we may consider bialgebras and Hopf algebras which are themselves objects in a braided category. A bialgebra in the category ${}^H\mathcal{M}$ is by definition a bialgebra in the usual sense, *i.e.* it obeys all of the usual axioms, but with its structure maps now as morphisms in the category. We call such an object K a *braided bialgebra*; we denote its structure maps by $(\underline{m}, \underline{\Delta}, \underline{\epsilon})$ if we wish to stress that they are morphisms in a braided category. In particular the coproduct $\underline{\Delta} : K \rightarrow K \underline{\otimes} K$ is required to be an algebra homomorphism from K into the braided tensor product. If K has also an antipode \underline{S} (obeying the usual axioms, but again required to intertwine the H -coaction) then we say that K is a *braided Hopf algebra*.

If K is a braided bialgebra in ${}^H\mathcal{M}$, we already mentioned that it becomes an H -module algebra *via* the canonical action (2.4). It follows that the vector space $K \otimes H$ may be equipped with the structure of an ‘ordinary’ bialgebra, given by the above cross product and cross coproduct constructions in (2.7) and (2.8) respectively. The resulting bialgebra is called the *cobosonisation* of K and is denoted $K \rtimes H$. A sufficient condition for $K \rtimes H$

to be a Hopf algebra is that K and H be Hopf algebras with the antipode of H invertible, in which case the antipode of $K \rtimes H$ is

$$(2.9) \quad S(a \otimes h) = (1 \otimes S^{-1}(a^{(-1)}h))(\underline{S}(a^{(0)}) \otimes 1).$$

The cobosonisation is of special interest because left $K \rtimes H$ -comodules are by construction exactly the same thing as left K -comodules in the category ${}^H\mathcal{M}$. The cobosonisation thus gives us the option of working either with a braided Hopf algebra K and its braided comodules or simply with comodules for the ordinary Hopf algebra $K \rtimes H$.

2.2. Cocycle deformations of Hopf algebras. We turn now to recalling the basic theory of quantisation of Hopf algebras by ‘cocycle cotwist’ as in [15]. Starting with a two-cocycle F on a Hopf algebra H , we give the relevant formulæ for obtaining a deformed Hopf algebra H_F as well as the appropriate deformations of the algebras or coalgebras on which H acts or coacts. We illustrate the theory with the well-known example of the noncommutative n -torus that we shall also use later on in the paper.

By a *two-cocycle* on a Hopf algebra H we mean a map $F : H \otimes H \rightarrow \mathbb{C}$ which is unital, convolution-invertible in the sense of (2.1) and obeys the cocycle condition $\partial F = 1$ or

$$(2.10) \quad F(g_{(1)}, f_{(1)})F(h_{(1)}, g_{(1)}f_{(2)})F^{-1}(h_{(2)}g_{(3)}, f_{(3)})F^{-1}(h_{(3)}, g_{(4)}) = \epsilon(f)\epsilon(h)\epsilon(g)$$

for all $f, g, h \in H$. Given such an F , there is a *cotwisted* Hopf algebra H_F which as a coalgebra is the same as H but whose product is replaced by

$$(2.11) \quad h \bullet_F g = F(h_{(1)}, g_{(1)})h_{(2)}g_{(2)}F^{-1}(h_{(3)}, g_{(3)})$$

and whose antipode becomes

$$(2.12) \quad S_F(h) := U(h_{(1)})S(h_{(2)})U^{-1}(h_{(3)}), \quad \text{with} \quad U(h) := F(h_{(1)}, Sh_{(2)}).$$

The cocycle condition (2.10) assures that the product in H_F is associative. If H has a coquasitriangular structure $\mathcal{R} : H \otimes H \rightarrow \mathbb{C}$, then H_F is also coquasitriangular with

$$(2.13) \quad \mathcal{R}_F(h, g) := F(g_{(1)}, h_{(1)})\mathcal{R}(h_{(2)}, g_{(2)})F^{-1}(h_{(3)}, g_{(3)}).$$

In the case where H is commutative, then \mathcal{R}_F in fact defines a cotriangular structure on H_F and, as a consequence, the induced braiding Ψ on the category ${}^{H_F}\mathcal{M}$ is symmetric in the sense that $\Psi^2 = \text{id}$.

In passing from H to H_F one finds that ${}^H\mathcal{M}$ and ${}^{H_F}\mathcal{M}$ are isomorphic as braided monoidal categories. Indeed, since the cotwist does not change the coalgebra structure of H , it follows that H -comodules are also H_F -comodules and H -comodule morphisms are also H_F -comodule morphisms; thus there is a functor $\mathcal{G}_F : {}^H\mathcal{M} \rightarrow {}^{H_F}\mathcal{M}$ which leaves the coactions unchanged. As categories, we have simply that ${}^H\mathcal{M}$ and ${}^{H_F}\mathcal{M}$ are just the same: the non-trivial part of the isomorphism is contained in what happens to the monoidal structure. Writing $V_F := \mathcal{G}_F(V)$, the category ${}^{H_F}\mathcal{M}$ gets a new monoidal structure by

$$\sigma_F : V_F \otimes W_F \rightarrow (V \otimes W)_F, \quad v \otimes w \mapsto F(v^{(-1)}, w^{(-1)})v^{(0)} \otimes w^{(0)}.$$

One checks [18] that \mathcal{G}_F is a monoidal functor and that it intertwines the braidings in ${}^H\mathcal{M}$ and ${}^{H_F}\mathcal{M}$ given respectively by \mathcal{R} and \mathcal{R}_F according to the formula (2.6). We call \mathcal{G}_F the ‘quantisation functor’ associated to the cocycle F since it simultaneously deforms all H -covariant constructions to corresponding versions which are covariant under H_F .

In particular, if A is an algebra in the category ${}^H\mathcal{M}$, then under the functor \mathcal{G}_F the product map $m : A \otimes A \rightarrow A$ becomes a map $(A \otimes A)_F \rightarrow A_F$. Composing this with σ_F yields a new product map

$$(2.14) \quad m_F : A_F \otimes A_F \rightarrow A_F, \quad a \otimes b \mapsto a \cdot_F b := F(a^{(-1)}, b^{(-1)})a^{(0)}b^{(0)},$$

and m_F automatically makes A_F into an H_F -comodule algebra.

In the same way, if A is a coalgebra in the category ${}^H\mathcal{M}$ with coproduct $\Delta : A \rightarrow A \otimes A$, applying the functor \mathcal{G}_F results in a map $A_F \rightarrow (A \otimes A)_F$. Then, composing with σ_F^{-1} yields a new coproduct map

$$(2.15) \quad \Delta_F : A_F \rightarrow A_F \otimes A_F, \quad a \mapsto F^{-1}(a_{(1)}^{(-1)}, a_{(2)}^{(-1)})a_{(1)}^{(0)} \otimes a_{(2)}^{(0)},$$

which automatically makes A_F into an H_F -comodule coalgebra with counit $\epsilon_F = \epsilon$.

We may of course put these two constructions together. Suppose that A is a bialgebra in the category ${}^H\mathcal{M}$, *i.e.* it is both an H -comodule algebra and an H -comodule coalgebra in a compatible way. Then we may simultaneously twist the product and coproduct on A and, as one might expect [17], the result is a bialgebra A_F in the category ${}^{H_F}\mathcal{M}$. Moreover, if A is a Hopf algebra in ${}^H\mathcal{M}$ with antipode \underline{S} , then A_F is a Hopf algebra in ${}^{H_F}\mathcal{M}$ with antipode $\underline{S}_F = \underline{S}$.

Remark 2.1. In the case where H is a Hopf $*$ -algebra, (thus in particular Δ is a $*$ -algebra map with $(S \circ *)^2 = \text{id}$), we need to add the condition that F is a *real cocycle* in the sense that

$$(2.16) \quad \overline{F(h, g)} = F((S^2 g)^*, (S^2 h)^*).$$

Then H_F acquires a deformed $*$ -structure

$$(2.17) \quad h^{*F} := \overline{V^{-1}(S^{-1}h_{(1)})(h_{(2)})^* V(S^{-1}h_{(3)})}, \quad \text{with} \quad V(h) := U^{-1}(h_{(1)})U(S^{-1}h_{(2)}).$$

Then, if A is a left H -comodule algebra and a $*$ -algebra such that the coaction is a $*$ -algebra map, the twisted algebra A_F gets a new $*$ -structure as well,

$$(2.18) \quad a^{*F} := \overline{V^{-1}(S^{-1}a^{(-1)})(a^{(0)})^*}.$$

Example 2.2. The Hopf algebra $H := \mathcal{A}(\mathbb{T}^n)$ of functions on the n -torus \mathbb{T}^n is the algebra

$$(2.19) \quad H := \mathcal{A}[t_j, t_j^{-1} \mid j = 1, \dots, n]$$

equipped with the Hopf $*$ -algebra structure

$$(2.20) \quad t_j^* = t_j^{-1}, \quad \Delta(t_j) = t_j \otimes t_j, \quad \epsilon(t_j) = 1, \quad S(t_j) = t_j^{-1}$$

for all $j = 1, \dots, n$, and, as usual, Δ , ϵ extended as $*$ -algebra maps and S extended as an anti- $*$ -algebra map. The canonical right action of \mathbb{T}^n on itself by group multiplication dualises to give a left coaction

$$(2.21) \quad \Delta_L : \mathcal{A}(\mathbb{T}^n) \rightarrow H \otimes \mathcal{A}(\mathbb{T}^n), \quad u_j \mapsto t_j \otimes u_j,$$

where we write $u_j, u_j^*, j = 1, \dots, n$, for the generators of $\mathcal{A}(\mathbb{T}^n)$ viewed as a left comodule algebra over itself. This coaction is equivalent to a grading of $\mathcal{A}(\mathbb{T}^n)$ by the Pontrjagin dual group \mathbb{Z}^n of \mathbb{T}^n , for which the homogeneous elements are the monomials of the form

$$(2.22) \quad t^{\vec{a}} := t_1^{a_1} t_2^{a_2} \dots t_n^{a_n}, \quad \vec{a} := (a_1, a_2, \dots, a_n) \in \mathbb{Z}^n.$$

One defines a two-cocycle F on H by choosing a real antisymmetric $n \times n$ matrix $\Theta = (\Theta_{jl})$ and setting

$$(2.23) \quad F(t^{\vec{a}}, t^{\vec{b}}) := \exp\left(i\pi(\vec{a} \cdot \Theta \cdot \vec{b})\right)$$

for homogeneous multi-degree elements $t^{\vec{a}}, t^{\vec{b}} \in H$ and extended by linearity. It is straightforward to verify that F is a cocycle which is real in the sense of Remark 2.1. Moreover, from its form as an exponential, this F is a Hopf bicharacter (cf. Eq. (2.2)), that is $F(fg, h) = F(f, h_{(1)})F(g, h_{(2)})$ and $F(f, gh) = F(f_{(1)}, h)F(f_{(2)}, g)$ for all $f, g, h \in H$. As a consequence it obeys

$$F(Sh, g) = F^{-1}(h, g), \quad F(h, Sg) = F^{-1}(h, g), \quad F(Sh, Sg) = F(h, g)$$

for all $g, h \in H$. These properties mean that F is determined by its values on the generators t_j , for which we have

$$(2.24) \quad F(t_j, t_l) = \exp(i\pi\Theta_{jl}), \quad j, l = 1, \dots, n.$$

The product, $*$ -structure and antipode on H are in fact undeformed by F , so $H = H_F$ as a Hopf $*$ -algebra. However, the trivial cotriangular structure $\mathcal{R} = \epsilon \otimes \epsilon$ of H twists into

$$\mathcal{R}_F(t_j, t_l) = F(t_l, t_j)F^{-1}(t_j, t_l) = F^{-2}(t_j, t_l).$$

As mentioned for the general construction, the deformation takes the form of an isomorphism of braided monoidal categories from ${}^H\mathcal{M}$ to ${}^{H_F}\mathcal{M}$. In particular, for $\mathcal{A}(\mathbb{T}^n)$, the effect is that, considered as left H -comodule algebra for itself, the $*$ -structure on $\mathcal{A}(\mathbb{T}^n)$ is unchanged but the product is twisted into a new product:

$$u_j \cdot_F u_l = u_j u_l F(t_j, t_l) = u_j u_l e^{i\pi\Theta_{jl}}.$$

We denote by $\mathcal{A}(\mathbb{T}_\Theta^n)$ the $*$ -algebra generated by the u_j, u_j^* with this new product; there are now relations

$$u_j \cdot_F u_l = e^{2i\pi\Theta_{jl}} u_l \cdot_F u_j, \quad u_l^* \cdot_F u_j = e^{2i\pi\Theta_{jl}} u_j \cdot_F u_l^*$$

for each pair of indices $j, l = 1, \dots, n$. The original torus \mathbb{T}^n has been quantised to give the noncommutative torus \mathbb{T}_Θ^n .

3. HOPF FIBRATIONS OVER SPHERES

In this section we review the construction in [9] of the $SU(2)$ -Hopf fibration over the noncommutative four-sphere S_θ^4 of [7]. We begin by giving the classical fibration in a coordinate algebra form, which we then quantise by means of a cocycle cotwist. We then review how the same twisting procedure also yields canonical differential calculi on the noncommutative algebras, as well as a Hodge operator $*_\theta$ on the sphere S_θ^4 .

3.1. The classical Hopf bundle. The coordinate algebra $\mathcal{A}(\mathbb{C}^4)$ of the vector space \mathbb{C}^4 is the associative commutative $*$ -algebra generated by the functions $z_j, j = 1, \dots, 4$, together with their conjugates $z_j^*, j = 1, \dots, 4$. The coordinate algebra $\mathcal{A}(S^7)$ of the seven-sphere is the quotient of $\mathcal{A}(\mathbb{C}^4)$ by the sphere relation

$$(3.1) \quad z_1^* z_1 + z_2^* z_2 + z_3^* z_3 + z_4^* z_4 = 1.$$

It is useful to arrange the generators of the algebra $\mathcal{A}(\mathbb{C}^4)$ into the matrix

$$(3.2) \quad u := \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ -z_2^* & z_1^* & -z_4^* & z_3^* \end{pmatrix}^t,$$

with t denoting matrix transposition, which we use to give a right action of the classical group $\mathrm{SU}(2)$ on $\mathcal{A}(\mathbb{C}^4)$ by

$$u \mapsto uw, \quad \text{with } w = \begin{pmatrix} w^1 & -\bar{w}^2 \\ w^2 & \bar{w}^1 \end{pmatrix} \in \mathrm{SU}(2).$$

This action preserves the sphere relation (3.1), whence it restricts to an action of $\mathrm{SU}(2)$ on the coordinate algebra $\mathcal{A}(S^7)$ of the seven-sphere. The subalgebra $\mathrm{Inv}_{\mathrm{SU}(2)}(\mathcal{A}(S^7))$ of invariant functions is generated as a commutative $*$ -algebra by the elements

$$(3.3) \quad \alpha = 2(z_1 z_3^* + z_2^* z_4), \quad \beta = 2(z_2 z_3^* - z_1^* z_4), \quad x = z_1 z_1^* + z_2 z_2^* - z_3 z_3^* - z_4 z_4^*,$$

together with their conjugates α^* , β^* and $x^* = x$. It follows from the sphere relation (3.1) that these generators obey the relation

$$(3.4) \quad \alpha^* \alpha + \beta^* \beta + x^2 = (z_1^* z_1 + z_2^* z_2 + z_3^* z_3 + z_4^* z_4)^2 = 1,$$

whence the invariant subalgebra is the coordinate algebra of a four-sphere,

$$\mathrm{Inv}_{\mathrm{SU}(2)}(\mathcal{A}(S^7)) = \mathcal{A}(S^4).$$

Indeed, the sphere relation (3.1) is equivalent to requiring that $u^* u = 1$, from which we automatically have that the matrix-valued function

$$(3.5) \quad q := uu^* = \frac{1}{2} \begin{pmatrix} 1+x & 0 & \alpha & -\beta^* \\ 0 & 1+x & \beta & \alpha^* \\ \alpha^* & \beta^* & 1-x & 0 \\ -\beta & \alpha & 0 & 1-x \end{pmatrix}$$

is a self-adjoint idempotent: $q^2 = q = q^*$. Clearly one has

$$(uw)(uw)^* = u(w w^*) u^* = uu^*, \quad w \in \mathrm{SU}(2),$$

and the entries of q really do generate an $\mathrm{SU}(2)$ -invariant subalgebra. Moreover, Eq. (3.3) defines an inclusion of algebras $\mathcal{A}(S^4) \hookrightarrow \mathcal{A}(S^7)$, which is just a coordinate algebra description of the standard Hopf fibration $S^7 \rightarrow S^4$ having $\mathrm{SU}(2)$ as structure group.

3.2. The noncommutative Hopf fibration. We now obtain a noncommutative version of the Hopf fibration using the method of ‘cocycle cotwisting’ as described in §2.2, with compatible torus (co)actions on the total and the base spheres. The ‘deforming’ Hopf algebra will be the algebra $H := \mathcal{A}(\mathbb{T}^2) = \mathcal{A}[t_j, t_j^* \mid j = 1, 2]$ of functions on the two-torus \mathbb{T}^2 with a ‘deforming’ two-cocycle F as given in (2.23). For the present case Θ is a real 2×2 antisymmetric matrix and hence of the form

$$\Theta = \frac{1}{2} \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix},$$

with $\theta \in \mathbb{R}$. We know from Example 2.2 that the twisted Hopf $*$ -algebra structure on $H = \mathcal{A}(\mathbb{T}^2)$ is in fact unchanged, so that $H = H_F$ as a Hopf $*$ -algebra, although the trivial cotriangular structure $\mathcal{R} = \epsilon \otimes \epsilon$ of H is twisted into $\mathcal{R}_F(t_j, t_l) := F^{-2}(t_j, t_l)$, leading to

a twisted product on H -comodule algebras. Indeed, we now use H_F to deform the Hopf fibration described in the previous section.

There is a left coaction of $H = \mathcal{A}(\mathbb{T}^2)$ on the coordinate algebra $\mathcal{A}(S^7)$ given by

$$(3.6) \quad \Delta_L : \mathcal{A}(S^7) \rightarrow H \otimes \mathcal{A}(S^7), \quad \Delta_L(z_j) = \tau_j \otimes z_j,$$

and extended as a $*$ -algebra map, where we write $(\tau_j) = (t_1, t_1^*, t_2, t_2^*)$ for the generators of $H = \mathcal{A}(\mathbb{T}^2)$. This coaction makes $\mathcal{A}(S^7)$ into a left H -comodule algebra, *i.e.* an algebra in the category ${}^H\mathcal{M}$. It follows that H also coacts on the four-sphere algebra $\mathcal{A}(S^4)$ by

$$(3.7) \quad \mathcal{A}(S^4) \rightarrow H \otimes \mathcal{A}(S^4), \quad x \mapsto 1 \otimes x, \quad \alpha \mapsto \tau_1 \tau_4 \otimes \alpha, \quad \beta \mapsto \tau_2 \tau_4 \otimes \beta,$$

making $\mathcal{A}(S^4)$ into an algebra in the category ${}^H\mathcal{M}$ as well.

Remark 3.1. The most general ‘toric’ (co)action on the sphere S^7 would be of a four-torus. We need to restrict to \mathbb{T}^2 in order to have actions which are compatible with the $SU(2)$ fibration. In fact, we are really dealing with a double cover $\tilde{\mathbb{T}}^2 \rightarrow \mathbb{T}^2$, with $\mathcal{A}(\tilde{\mathbb{T}}^2)$ coacting on $\mathcal{A}(S^7)$ and $\mathcal{A}(\mathbb{T}^2)$ coacting on $\mathcal{A}(S^4)$, as is clear from Eqs. (3.6) and (3.7). We shall take the liberty of being sloppy on this point here and in the following.

The product on $\mathcal{A}(S^7)$ is then deformed by comodule cotwist (*cf.* §2.2) into

$$z_j \cdot_F z_l = F(\tau_j, \tau_l) z_j z_l, \quad z_j \cdot_F z_l^* = F(\tau_j, \tau_l^*) z_j z_l^*.$$

Introducing the deformation parameter $\eta_{jl} = \mathcal{R}_F(\tau_j, \tau_l) = F^{-2}(\tau_j, \tau_l)$ given explicitly by

$$(3.8) \quad (\eta_{jl}) = \begin{pmatrix} 1 & 1 & \mu & \bar{\mu} \\ 1 & 1 & \bar{\mu} & \mu \\ \bar{\mu} & \mu & 1 & 1 \\ \mu & \bar{\mu} & 1 & 1 \end{pmatrix}, \quad \mu = e^{i\pi\theta},$$

the deformed algebra relations are computed to be (dropping the product symbol \cdot_F)

$$z_j z_l = \eta_{lj} z_l z_j, \quad z_j z_l^* = \eta_{jl} z_l^* z_j.$$

We denote by $\mathcal{A}(S_\theta^7)$ the algebra generated by $\{z_j, z_j^* \mid j = 1, \dots, 4\}$ modulo these relations. In this way, $\mathcal{A}(S_\theta^7)$ is an algebra in the category ${}^{H_F}\mathcal{M}$ of left H_F -comodules.

Similarly, the product on $\mathcal{A}(S^4)$ is twisted into

$$\alpha \cdot_F \beta = F(\tau_1 \tau_4, \tau_2 \tau_4) \alpha \beta, \quad \alpha \cdot_F \beta^* = F(\tau_1 \tau_4, \tau_2^* \tau_4^*) \alpha \beta^*.$$

With deformation parameter $\lambda := \mu^2 = e^{i2\pi\theta}$, the algebra relations become (again dropping the product symbol \cdot_F)

$$\alpha \beta = \lambda \beta \alpha, \quad \alpha^* \beta^* = \lambda \beta^* \alpha^*, \quad \beta^* \alpha = \lambda \alpha \beta^*, \quad \beta \alpha^* = \lambda \alpha^* \beta,$$

with x central. We denote by $\mathcal{A}(S_\theta^4)$ the algebra generated by α, β, x and their conjugates, subject to these relations. They make $\mathcal{A}(S_\theta^4)$ into an algebra in the category ${}^{H_F}\mathcal{M}$.

Since the coaction of H on $\mathcal{A}(S^7)$ commutes with the $SU(2)$ -action (3.6), the deformation of the spheres $\mathcal{A}(S^7)$ and $\mathcal{A}(S^4)$ preserves this action and hence there is an algebra inclusion $\mathcal{A}(S_\theta^4) \hookrightarrow \mathcal{A}(S_\theta^7)$, giving a noncommutative principal bundle with classical structure group $SU(2)$. As mentioned, the above H -coaction is the only one which is compatible

with the bundle structure [9]. The elements α, β, x and their adjoints are the entries of the projection \mathfrak{q} which is now given by

$$(3.9) \quad \mathfrak{q} := uu^* = \frac{1}{2} \begin{pmatrix} 1+x & 0 & \alpha & -\bar{\mu}\beta^* \\ 0 & 1+x & \beta & \mu\alpha^* \\ \alpha^* & \beta^* & 1-x & 0 \\ -\mu\beta & \bar{\mu}\alpha & 0 & 1-x \end{pmatrix};$$

note that the matrix u is still of the form in Eq. (3.2).

3.3. Noncommutative differential calculi. There are canonical differential structures on each of the spheres $\mathcal{A}(S_\theta^7)$ and $\mathcal{A}(S_\theta^4)$ as deformations of their classical counterparts. They are constructed as follows.

We begin with the space $\mathcal{A}(\mathbb{C}^4)$. Let $\Omega(\mathbb{C}^4)$ be the usual differential calculus on $\mathcal{A}(\mathbb{C}^4)$, generated as a commutative differential graded algebra by the degree zero elements z_j, z_l^* and degree one elements dz_j, dz_l^* , satisfying the relations

$$dz_j \wedge dz_l + dz_l \wedge dz_j = 0, \quad dz_j \wedge dz_l^* + dz_l^* \wedge dz_j = 0,$$

for $j, l = 1, \dots, 4$. The differential d is defined by $z_j \mapsto dz_j, z_l^* \mapsto dz_l^*$ and extended uniquely using a graded Leibniz rule. The coaction Δ_L of Eq. (3.6) on $\mathcal{A}(\mathbb{C}^4)$ extends to one on the differential calculus $\Omega(\mathbb{C}^4)$ by defining it to commute with the differential d . We may therefore deform the differential structure $\Omega(\mathbb{C}^4)$ in the same way as we did for the algebra itself, by comodule cotwist:

$$(3.10) \quad z_j \cdot_F dz_l = F(\tau_j, \tau_l) z_l dz_j, \quad z_j \cdot_F dz_l^* = F(\tau_j, \tau_l^*) z_l^* dz_j, \\ dz_j \wedge_F dz_l = F(\tau_j, \tau_l) dz_j \wedge dz_l.$$

There is hence a canonical differential graded algebra $\Omega(\mathbb{C}_\theta^4)$ for $\mathcal{A}(\mathbb{C}_\theta^4)$, with the same generators but now subject to the relations (again no explicit deformed product symbol)

$$z_j dz_l = \eta_{lj} (dz_l) z_j, \quad z_j dz_l^* = \eta_{jl} (dz_l^*) z_j, \\ dz_j \wedge dz_l + \eta_{lj} dz_l \wedge dz_j = 0, \quad dz_j \wedge dz_l^* + \eta_{jl} dz_l^* \wedge dz_j = 0$$

for $j, l = 1, \dots, 4$. Note that the relations in the twisted calculus $\Omega(\mathbb{C}_\theta^4)$ are the same as those for the coordinate algebra $\mathcal{A}(\mathbb{C}_\theta^4)$, but now with d inserted.

Since the differential d is undeformed, the same strategy also defines a differential calculus $\Omega(S_\theta^7)$ on $\mathcal{A}(S_\theta^7)$ as a cotwist of the classical one, with the products between generators also given by Eq. (3.10). Similarly, one obtains a differential calculus $\Omega(S_\theta^4)$ on the four-sphere $\mathcal{A}(S_\theta^4)$. It is generated by the degree zero elements $\alpha, \alpha^*, \beta, \beta^*, x$ and the degree one elements $d\alpha, d\alpha^*, d\beta, d\beta^*, dx$, obeying relations as for the coordinate algebra $\mathcal{A}(S_\theta^4)$ but now with d inserted, namely

$$\alpha d\beta = \lambda d\beta \alpha, \quad \beta^* d\alpha = \lambda d\alpha \beta^*, \quad d\alpha d\beta + \lambda d\beta d\alpha = 0$$

and so on, with x and dx obeying the same undeformed relations as in the classical case. The calculus $\Omega(S_\theta^4)$ may be obtained either as the $SU(2)$ -invariant part of $\Omega(S_\theta^7)$ or directly as a comodule cotwist of its classical counterpart.

The torus \mathbb{T}^2 acts on the sphere S^4 by isometries and hence leaves the conformal structure invariant. As a consequence, one checks that the classical Hodge operator (in

particular on two-forms) $*$: $\Omega^2(S^4) \rightarrow \Omega^2(S^4)$ is an intertwiner for the coaction Δ_L of the torus algebra $H = \mathcal{A}(\mathbb{T}^2)$, that is to say

$$\Delta_L(*\omega) = (\text{id} \otimes *)\Delta_L(\omega), \quad \omega \in \Omega^2(S^4).$$

Since the deformed differential calculus $\Omega(S_\theta^4)$ coincides as a vector space with its undeformed counterpart $\Omega(S^4)$, we can define a Hodge operator $*_\theta$ on $\Omega(S_\theta^4)$ by the same formula as the classical $*$, yielding a map $*_\theta : \Omega^2(S_\theta^4) \rightarrow \Omega^2(S_\theta^4)$ which is by construction a morphism in the category ${}^{H_F}\mathcal{M}$. This is all we need when studying instantons on S_θ^4 .

4. BRAIDED MATRIX ALGEBRAS

The previous section constructed the coordinate algebras of the noncommutative spaces \mathbb{C}_θ^4 , S_θ^7 and S_θ^4 as objects in the category of left H_F -comodules. In this section we observe that the various matrix algebras which act upon these spaces may also be naturally thought of as objects in the same category and, as a result, they are obtained using exactly the same ‘quantisation’ procedure.

4.1. The classical groups $\text{SL}(2, \mathbb{H})$ and $\text{Sp}(2)$. We denote by $M(2, \mathbb{H})$ the algebra of 2×2 matrices with quaternion entries; for convenience we shall write them as 4×4 matrices with complex entries. The classical bialgebra $\mathcal{A}(M(2, \mathbb{H}))$ of functions on $M(2, \mathbb{H})$ is defined to be the commutative associative algebra generated by the coordinate functions arranged in the following 4×4 matrix

$$(4.1) \quad A = \begin{pmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{pmatrix} = \begin{pmatrix} a_1 & -a_2^* & b_1 & -b_2^* \\ a_2 & a_1^* & b_2 & b_1^* \\ c_1 & -c_2^* & d_1 & -d_2^* \\ c_2 & c_1^* & d_2 & d_1^* \end{pmatrix}.$$

We think of this matrix as being generated by a set of quaternion-valued functions, writing

$$a = (a_{ij}) = \begin{pmatrix} a_1 & -a_2^* \\ a_2 & a_1^* \end{pmatrix}$$

and similarly for the other entries b, c, d . The $*$ -structure on this algebra is evident from the matrix (4.1). We equip $\mathcal{A}(M(2, \mathbb{H}))$ with the matrix bialgebra structure

$$\Delta(A_{ij}) = \sum_{\alpha} A_{i\alpha} \otimes A_{\alpha j}, \quad \epsilon(A_{ij}) = \delta_{ij} \quad \text{for } i, j = 1, \dots, 4.$$

Of course, it is not a Hopf algebra since it does not have an antipode (this is equivalent to saying that the matrix algebra $M(2, \mathbb{H})$ is not quite a group). We obtain a Hopf algebra by passing to the quotient of $\mathcal{A}(M(2, \mathbb{H}))$ by the Hopf $*$ -ideal generated by the element $D - 1$, where $D = \det(A)$ is the determinant of the matrix A . We denote the quotient by $\mathcal{A}(\text{SL}(2, \mathbb{H}))$, the algebra of functions on the group $\text{SL}(2, \mathbb{H})$ of matrices in $M(2, \mathbb{H})$ with determinant one. The algebra $\mathcal{A}(\text{SL}(2, \mathbb{H}))$ inherits a $*$ -bialgebra structure from that of $\mathcal{A}(M(2, \mathbb{H}))$ and we define an antipode by

$$S(A_{ij}) = (-1)^{i+j} A'_{ji},$$

with

$$(4.2) \quad A'_{ij} := \sum_{\sigma \in S_3} (-1)^{|\sigma|} \epsilon^{\sigma_1 \dots \sigma_{i-1} l \sigma_{i+1} \dots \sigma_4} A_{1, \sigma_1} \dots A_{i-1, \sigma_{i-1}} A_{i+1, \sigma_{i+1}} \dots A_{4, \sigma_4},$$

and ϵ^{ijkl} is the alternating symbol on four elements. The notation is

$$(\sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_4) = \sigma(1, \dots, l-1, l+1, \dots, 4).$$

with σ an element of S_3 , the permutation group on three objects (once an index is fixed the remaining one can take only three possible values).

Definition 4.1. The datum $\mathcal{A}(\mathrm{SL}(2, \mathbb{H})) = (\mathcal{A}(\mathrm{SL}(2, \mathbb{H})), \Delta, \epsilon, S)$ constitutes a Hopf algebra. We define the Hopf algebra $\mathcal{A}(\mathrm{Sp}(2))$ to be the quotient of $\mathcal{A}(\mathrm{SL}(2, \mathbb{H}))$ by the two-sided $*$ -Hopf ideal \mathcal{I} generated by elements

$$(4.3) \quad \sum_{\alpha} (A^*)_{i\alpha} A_{\alpha j} - \delta_{ij}, \quad i, j = 1, \dots, 4.$$

In the algebra $\mathcal{A}(\mathrm{Sp}(2))$ there are relations $A^*A = AA^* = 1$, or equivalently $S(A) = A^*$.

4.2. The braided groups $\mathcal{B}(\mathrm{SL}_{\theta}(2, \mathbb{H}))$ and $\mathcal{B}(\mathrm{Sp}_{\theta}(2))$. There is an embedding of a two-torus \mathbb{T}^2 into $\mathrm{M}(2, \mathbb{H})$ as a diagonal subgroup, given by the map

$$(4.4) \quad \rho : \mathbb{T}^2 \rightarrow \mathrm{M}(2, \mathbb{H}), \quad \rho(s) = \mathrm{diag}(e^{2\pi i s_1}, e^{2\pi i s_2}),$$

where $s = (e^{2\pi i s_1}, e^{2\pi i s_2}) \in \mathbb{T}^2$. At the level of coordinate algebras, this inclusion becomes a bialgebra projection

$$(4.5) \quad \pi : \mathcal{A}(\mathrm{M}(2, \mathbb{H})) \rightarrow \mathcal{A}(\mathbb{T}^2), \quad \pi(A_{ij}) = \delta_{ij} \tau_j,$$

where $(\tau_j) = (t_1, t_1^*, t_2, t_2^*)$ in terms of the generators of $H = \mathcal{A}(\mathbb{T}^2)$. A resulting right adjoint action of \mathbb{T}^2 on $\mathrm{M}(2, \mathbb{H})$ given by

$$\mathrm{M}(2, \mathbb{H}) \times \mathbb{T}^2 \rightarrow \mathrm{M}(2, \mathbb{H}), \quad (g, s) \mapsto \rho(s^{-1}) \cdot g \cdot \rho(s),$$

in turn dualises to the left H -adjoint coaction given by

$$(4.6) \quad \mathrm{Ad}_L : \mathcal{A}(\mathrm{M}(2, \mathbb{H})) \rightarrow H \otimes \mathcal{A}(\mathrm{M}(2, \mathbb{H})), \quad \mathrm{Ad}_L(A_{ij}) = \tau_i \tau_j^* \otimes A_{ij}$$

for $i, j = 1, \dots, 4$ and extended as a $*$ -algebra map. This coaction realises $\mathcal{A}(\mathrm{M}(2, \mathbb{H}))$ as an object in the category ${}^H\mathcal{M}$ of left H -comodules. Since the algebra is commutative, it follows that its product is a morphism in the category,

$$\mathrm{Ad}_L(A_{ij} A_{kl}) = \mathrm{Ad}_L(A_{ij}) \mathrm{Ad}_L(A_{kl}).$$

The fact that the adjoint action preserves matrix multiplication in $\mathrm{M}(2, \mathbb{H})$ means that the coproduct on $\mathcal{A}(\mathrm{M}(2, \mathbb{H}))$ is covariant under the coaction Ad_L , *i.e.* $\mathcal{A}(\mathrm{M}(2, \mathbb{H}))$ is an H -comodule coalgebra. The same is true of the counit, whence $\mathcal{A}(\mathrm{M}(2, \mathbb{H}))$ is a bialgebra in the category ${}^H\mathcal{M}$ of left H -comodules. Similarly the antipodes on $\mathcal{A}(\mathrm{SL}(2, \mathbb{H}))$ and $\mathcal{A}(\mathrm{Sp}(2))$ respect the H -coaction and so they form Hopf algebras in the category ${}^H\mathcal{M}$.

What happens to this picture under cotwisting is clear. We know from Example 2.2 that upon twisting the torus algebra $H = \mathcal{A}(\mathbb{T}^2)$ with a twist F like the one in §3.2, as a Hopf $*$ -algebra $H_F = H$, although the coquasitriangular structure twists and the deformation takes the form of a ‘quantisation functor’, *i.e.* an isomorphism of braided monoidal categories from ${}^H\mathcal{M}$ to ${}^{H_F}\mathcal{M}$. This functor leaves objects and coactions unchanged, and hence the adjoint coaction (4.6) also realises $\mathcal{A}(\mathrm{M}(2, \mathbb{H}))$ as an object in the category ${}^{H_F}\mathcal{M}$. However, we need to deform the bialgebra structures (the product and coproduct) on $\mathcal{A}(\mathrm{M}(2, \mathbb{H}))$ in order to maintain covariance.

Using Eq. (2.14), the product is deformed into

$$(4.7) \quad A_{ij} \dot{\cdot} A_{kl} = F(\tau_i \tau_j^*, \tau_k \tau_l^*) A_{ij} A_{kl},$$

denoting the twisted product by $\dot{\cdot}$ in order to stress its being a morphism in a braided category. We write $\mathcal{B}(\mathcal{M}_\theta(2, \mathbb{H}))$ for the algebra generated by the A_{ij} , $i, j = 1, \dots, 4$, equipped with the twisted product. Likewise, using Eq. (2.15) the coproduct is deformed on generators into

$$(4.8) \quad \Delta_F(A_{ij}) = \sum_{\alpha} A_{i\alpha} \otimes A_{\alpha j} F^{-1}(\tau_i \tau_{\alpha}^*, \tau_{\alpha} \tau_j^*)$$

and extended then as an algebra homomorphism to the braided tensor product,

$$(4.9) \quad \Delta_F : \mathcal{B}(\mathcal{M}_\theta(2, \mathbb{H})) \rightarrow \mathcal{B}(\mathcal{M}_\theta(2, \mathbb{H})) \underline{\otimes} \mathcal{B}(\mathcal{M}_\theta(2, \mathbb{H})).$$

If one defines a new set of generators of $\mathcal{B}(\mathcal{M}_\theta(2, \mathbb{H}))$ by

$$(4.10) \quad \hat{A}_{ij} := F^{-1}(\tau_i, \tau_j) A_{ij},$$

then with respect to these generators the coproduct has the standard matrix form

$$\Delta_F(\hat{A}_{ij}) = \sum_{\alpha} \hat{A}_{i\alpha} \otimes \hat{A}_{\alpha j}.$$

In order to obtain a braided bialgebra, it is necessary to have Δ_F respect the algebra structure of $\mathcal{B}(\mathcal{M}_\theta(2, \mathbb{H}))$: the fact that it does so is a consequence of the dual version of [17, Thm 2.8], although we can prove it directly as follows. We first note that, using the Hopf bicharacter property of F , one has

$$F(\tau_i \tau_{\alpha}^*, \tau_{\alpha} \tau_j^*) = F(\tau_i, \tau_{\alpha}) F^{-1}(\tau_i, \tau_j) F^{-1}(\tau_{\alpha}, \tau_{\alpha}) F(\tau_{\alpha}, \tau_j).$$

Lemma 4.2. *The coproduct Δ_F and the product $\dot{\cdot}$ make $\mathcal{B}(\mathcal{M}_\theta(2, \mathbb{H}))$ into a bialgebra in the category ${}^{H_F}\mathcal{M}$ of left H_F -comodules.*

Proof. By construction, the vector space $\mathcal{B}(\mathcal{M}_\theta(2, \mathbb{H}))$ equipped with the product $\dot{\cdot}$ and the coproduct Δ_F is certainly both an algebra and a coalgebra in the category ${}^{H_F}\mathcal{M}$ via the left adjoint coaction. Using the product $\dot{\cdot}$ and the braiding in the category, we now compute that

$$\begin{aligned} \Delta_F(\hat{A}_{ij} \dot{\cdot} \hat{A}_{kl}) &= \Delta_F(\hat{A}_{ij} \hat{A}_{kl}) F(\tau_i \tau_j^*, \tau_k \tau_l^*) \\ &= \sum_{\alpha, \beta} \hat{A}_{i\alpha} \hat{A}_{k\beta} \otimes \hat{A}_{\alpha j} \hat{A}_{\beta l} F^{-1}(\tau_i \tau_{\alpha}^* \tau_k \tau_{\beta}^*, \tau_{\alpha} \tau_j^* \tau_{\beta} \tau_l^*) F(\tau_i \tau_j^*, \tau_k \tau_l^*) \\ &= \sum_{\alpha, \beta} \hat{A}_{i\alpha} \dot{\cdot} \hat{A}_{k\beta} \otimes \hat{A}_{\alpha j} \dot{\cdot} \hat{A}_{\beta l} F^{-1}(\tau_i \tau_{\alpha}^* \tau_k \tau_{\beta}^*, \tau_{\alpha} \tau_j^* \tau_{\beta} \tau_l^*) \times \\ &\quad \times F(\tau_i \tau_j^*, \tau_k \tau_l^*) F^{-1}(\tau_i \tau_{\alpha}^*, \tau_k \tau_{\beta}^*) F^{-1}(\tau_{\alpha} \tau_j^*, \tau_{\beta} \tau_l^*) \\ &= \sum_{\alpha, \beta} \hat{A}_{i\alpha} \dot{\cdot} \hat{A}_{k\beta} \otimes \hat{A}_{\alpha j} \dot{\cdot} \hat{A}_{\beta l} F^{-2}(\tau_k \tau_{\beta}^*, \tau_{\alpha} \tau_j^*) \\ &= \Delta_F(\hat{A}_{ij}) \dot{\cdot} \Delta_F(\hat{A}_{kl}), \end{aligned}$$

where in the first equality we have used the definition of the twisted product, in the second equality we have applied the definition of Δ_F and in the third equality we have again used the definition of the product. The fourth equality involves a simplification of the terms in F using its Hopf bicharacter properties. The coproduct Δ_F is therefore an algebra map with respect to the twisted product, whence the result. \square

In the same way, it follows that there are bialgebras $\mathcal{B}(\mathrm{SL}_\theta(2, \mathbb{H}))$ and $\mathcal{B}(\mathrm{Sp}_\theta(2))$ in the category ${}^{H_F}\mathcal{M}$, obtained by comodule cotwist of the classical bialgebras $\mathcal{A}(\mathrm{SL}(2, \mathbb{H}))$ and $\mathcal{A}(\mathrm{Sp}(2))$. The same formula as in Eq. (4.1), although now using the braided product, defines an antipode for $\mathcal{B}(\mathrm{SL}_\theta(2, \mathbb{H}))$ which we now denote by \underline{S} . The antipode is extended as a morphism in the category ${}^{H_F}\mathcal{M}$, namely as a braided anti-algebra map

$$\underline{S} \circ \cdot = \cdot \circ (\underline{S} \otimes \underline{S}) \circ \Psi,$$

where Ψ is the braiding in the category of left H_F -comodules, thus making $\mathcal{B}(\mathrm{SL}_\theta(2, \mathbb{H}))$ into a braided Hopf algebra. Similarly, the formula

$$\underline{S}(A_{ij}) = (A^*)_{ji},$$

extended as a braided anti-algebra map, defines an antipode on $\mathcal{B}(\mathrm{Sp}_\theta(2))$, also making it into a braided Hopf algebra in the category.

5. BRAIDED SYMMETRIES OF NONCOMMUTATIVE SPHERES

Now that we have constructed braided versions of the transformation algebra $\mathrm{M}(2, \mathbb{H})$ and its various quotients, we are able to show how they (co)act upon the spaces \mathbb{C}_θ^4 , S_θ^7 and S_θ^4 given in §3. The important technical point that we illustrate is that these coactions are themselves morphisms in the category ${}^{H_F}\mathcal{M}$ and so necessarily braided. We then construct the cobosonisations of the transformation algebras using the procedure described in §2.1, the advantage of doing so being, as mentioned, that it takes us from the realm of braided geometry back into the realm of ‘ordinary’ quantum groups.

5.1. Symmetries in the braided category. Recall that we arranged the generators of the algebra $\mathcal{A}(\mathbb{C}^4)$ into the 4×2 matrix $u = (u_{ia})$ for $i = 1, \dots, 4$ and $a = 1, 2$, given in Eq. (3.2). In this notation the H -coaction can be written

$$\mathcal{A}(\mathbb{C}^4) \rightarrow H \otimes \mathcal{A}(\mathbb{C}^4), \quad u_{ia} \mapsto \tau_i \otimes u_{ia}.$$

A left coaction of the classical bialgebra $\mathcal{A}(\mathrm{M}(2, \mathbb{H}))$ on $\mathcal{A}(\mathbb{C}^4)$ is given on generators by

$$(5.1) \quad \mathcal{A}(\mathbb{C}^4) \rightarrow \mathcal{A}(\mathrm{M}(2, \mathbb{H})) \otimes \mathcal{A}(\mathbb{C}^4), \quad u_{ia} \mapsto \sum_\beta A_{i\beta} \otimes u_{\beta a},$$

for $i = 1, \dots, 4$, $a = 1, 2$ and extended as a $*$ -algebra map. It is clear that this coaction is a morphism in the category ${}^H\mathcal{M}$, which becomes a morphism in ${}^{H_F}\mathcal{M}$ upon applying the quantisation functor. As discussed in §2.2, the coaction itself does not change, but we must remember that the monoidal structure is deformed. In this way, we get a left coaction of the braided bialgebra $\mathcal{B}(\mathrm{M}_\theta(2, \mathbb{H}))$ on $\mathcal{A}(\mathbb{C}_\theta^4)$, given on generators by

$$(5.2) \quad \Delta_L : \mathcal{A}(\mathbb{C}_\theta^4) \rightarrow \mathcal{B}(\mathrm{M}_\theta(2, \mathbb{H})) \underline{\otimes} \mathcal{A}(\mathbb{C}_\theta^4), \quad u_{ia} \mapsto \sum_\beta \hat{A}_{i\beta} \otimes u_{\beta a}.$$

As the notation suggests, this coaction extends as an algebra homomorphism to the braided tensor product, so that we have, for example,

$$u_{ia} u_{lb} \mapsto \sum_{\beta, \gamma} (\hat{A}_{i\beta} \otimes u_{\beta a}) \cdot (\hat{A}_{l\gamma} \otimes u_{\gamma b}) = \sum_{\beta, \gamma} \hat{A}_{i\beta} \cdot \hat{A}_{l\gamma} \otimes u_{\beta a} u_{\gamma b} F^{-2}(\tau_l \tau_\gamma^*, \tau_\beta),$$

with $i, l = 1, \dots, 4$ and $a, b = 1, 2$. This argument also applies to the braided Hopf algebra $\mathcal{B}(\mathrm{Sp}_\theta(2))$, yielding a left coaction given by a similar expression:

$$(5.3) \quad \Delta_L : \mathcal{A}(\mathbb{C}_\theta^4) \rightarrow \mathcal{B}(\mathrm{Sp}_\theta(2)) \underline{\otimes} \mathcal{A}(\mathbb{C}_\theta^4), \quad u_{ia} \mapsto \sum_\beta \hat{A}_{i\beta} \otimes u_{\beta a}.$$

Now, this second coaction preserves the sphere relation of S_θ^7 , since we have

$$\begin{aligned} \sum_\alpha z_\alpha^* z_\alpha &\mapsto \sum_{\alpha, \beta, \gamma} \left((1 \otimes z_\beta^*) \cdot ((\hat{A}_{\alpha\beta})^* \otimes 1) \right) \cdot (\hat{A}_{\alpha\gamma} \otimes z_\gamma) \\ &= \sum_{\alpha, \beta, \gamma} (A^*)_{\beta\alpha} \cdot A_{\alpha\gamma} \otimes z_\beta^* z_\gamma F^{-2}(\tau_\alpha^* \tau_\beta, \tau_\beta^*) F^{-2}(\tau_\alpha \tau_\gamma^*, \tau_\beta^*) \\ &= \sum_{\alpha, \beta, \gamma} (\hat{A}^*)_{\beta\alpha} \cdot \hat{A}_{\alpha\gamma} \otimes z_\beta^* z_\gamma F^{-2}(\tau_\gamma, \tau_\beta) \\ &= \sum_{\beta, \gamma} \delta_{\beta\gamma} \otimes z_\beta^* z_\gamma F^{-2}(\tau_\gamma, \tau_\beta) = \sum_\beta 1 \otimes z_\beta^* z_\beta, \end{aligned}$$

and thus it descends to a coaction on the sphere $\mathcal{A}(S_\theta^7)$,

$$(5.4) \quad \Delta_L : \mathcal{A}(S_\theta^7) \rightarrow \mathcal{B}(\mathrm{Sp}_\theta(2)) \underline{\otimes} \mathcal{A}(S_\theta^7),$$

defined by the same formula. Similarly, it is straightforward to check that the $\mathcal{B}(\mathrm{Sp}_\theta(2))$ -coaction restricts to the subalgebra $\mathcal{A}(S_\theta^4)$ generated by the entries of the projection $\mathbf{q} = uu^*$ of Eq. (3.9). The entries of \mathbf{q} are of the form $\mathbf{q}_{kl} := (uu^*)_{kl} = \sum_a u_{ka}(u^*)_{al}$ for $k, l = 1, \dots, 4$, so that the coaction has the form

$$\begin{aligned} (5.5) \quad \Delta_L : \mathcal{A}(S_\theta^4) &\rightarrow \mathcal{B}(\mathrm{Sp}_\theta(2)) \underline{\otimes} \mathcal{A}(S_\theta^4), \\ \Delta_L(\mathbf{q}_{kl}) &= \sum_{\beta, \gamma, a} \left(\hat{A}_{k\beta} \otimes u_{\beta a} \right) \cdot \left((\hat{A}_{l\gamma})^* \otimes (u_{\gamma a})^* F^{-2}(\tau_\gamma \tau_l^*, \tau_\gamma^*) \right) \\ &= \sum_{\beta, \gamma, a} \hat{A}_{k\beta} \cdot (\hat{A}^*)_{\gamma l} \otimes u_{\beta a} (u^*)_{a\gamma} F^{-2}(\tau_\gamma \tau_l^*, \tau_\gamma^*) F^{-2}(\tau_\gamma \tau_l^*, \tau_\beta) \\ &= \sum_{\beta, \gamma} \hat{A}_{k\beta} \cdot (\hat{A}^*)_{\gamma l} \otimes \mathbf{q}_{\beta\gamma} F^{-2}(\tau_\gamma \tau_l^*, \tau_\beta \tau_\gamma^*), \end{aligned}$$

for each pair of indices $k, l = 1, \dots, 4$.

5.2. Braided conformal transformations. The story is similar for obtaining a coaction of $\mathcal{B}(\mathrm{SL}_\theta(2, \mathbb{H}))$ on the quantum spheres, albeit slightly more complicated. The formula (5.2) also defines a left coaction Δ_L of $\mathcal{B}(\mathrm{SL}_\theta(2, \mathbb{H}))$ on the algebra $\mathcal{A}(\mathbb{C}_\theta^4)$ although, just as in the classical case, it does not preserve the sphere relation in $\mathcal{A}(S_\theta^7)$. Here, the effect of the coaction is to ‘inflate’ the spheres, *i.e.* it maps the element $r^2 := \sum_\alpha z_\alpha^* z_\alpha$ to

$$\rho^2 := \sum_\alpha \Delta_L(z_\alpha^* z_\alpha),$$

which is not equal to $1 \otimes \sum_\alpha z_\alpha^* z_\alpha$ in the algebra $\Delta_L(\mathcal{A}(\mathbb{C}_\theta^4))$. Since r^2 is self-adjoint and central in $\mathcal{A}(\mathbb{C}_\theta^4)$, we may evaluate it as a positive real number to give the coordinate algebra of a noncommutative sphere of fixed radius r . As this radius varies in $\mathcal{A}(\mathbb{C}_\theta^4)$, it sweeps out a family of seven-spheres. Similarly, we may evaluate the central element ρ^2 in $\mathcal{A}(\tilde{\mathbb{C}}_\theta^4) := \Delta_L(\mathcal{A}(\mathbb{C}_\theta^4)) \subset \mathcal{B}(\mathrm{SL}_\theta(2, \mathbb{H})) \underline{\otimes} \mathcal{A}(\mathbb{C}_\theta^4)$ to obtain the coordinate algebra of a noncommutative sphere of fixed radius ρ : as the value of ρ varies in $\mathcal{A}(\tilde{\mathbb{C}}_\theta^4)$ it sweeps out another family of seven-spheres. The effect of the coaction Δ_L of $\mathcal{B}(\mathrm{SL}_\theta(2, \mathbb{H}))$ is to map the former family onto the latter. We introduce the notation $\mathcal{A}_r(S_\theta^7) := \mathcal{A}(\mathbb{C}_\theta^4)$ and $\mathcal{A}_\rho(\tilde{S}_\theta^7) := \mathcal{A}(\tilde{\mathbb{C}}_\theta^4)$, which does nothing other than to stress the fact that we think of the spaces \mathbb{C}_θ^4 and $\tilde{\mathbb{C}}_\theta^4$ as families of quantum seven-spheres parameterised by the values of the functions r^2 and ρ^2 , respectively.

We find a similar picture for the four-sphere S_θ^4 . The coaction of $\mathcal{B}(\mathrm{SL}_\theta(2, \mathbb{H}))$ does not preserve the sphere relation, but rather gives

$$\Delta_L(\alpha\alpha^* + \beta\beta^* + x^2) = \rho^4,$$

whence S_θ^4 is also ‘inflated’ by the action of $\mathrm{SL}_\theta(2, \mathbb{H})$. Writing $\mathcal{A}_r(S_\theta^4)$ for the $*$ -subalgebra of $\mathcal{A}(\mathbb{C}_\theta^4)$ generated by α, β, x and their conjugates, then as r^4 varies it sweeps out a family of noncommutative four-spheres. Evaluating r^4 as a real number yields the coordinate algebra of a noncommutative four-sphere of radius r^2 .

Similarly, we define $\tilde{\alpha} := \Delta(\alpha)$, $\tilde{\beta} := \Delta_L(\beta)$, $\tilde{x} := \Delta_L(x)$ and so forth, writing $\mathcal{A}_\rho(\tilde{S}_\theta^4)$ for the $*$ -subalgebra of $\mathcal{A}(\tilde{\mathbb{C}}_\theta^4)$ that they generate. As the value of ρ^4 varies in $\mathcal{A}(\tilde{\mathbb{C}}_\theta^4)$, it sweeps out a family of noncommutative four-spheres of radius ρ^2 . The effect of the coaction Δ_L is to map the former family onto the latter.

The algebra $\mathcal{A}_r(S_\theta^4)$ is precisely the $\mathrm{SU}(2)$ -invariant subalgebra of $\mathcal{A}_r(S_\theta^7)$ and $\mathcal{A}_\rho(\tilde{S}_\theta^4)$ is the $\mathrm{SU}(2)$ -invariant subalgebra of $\mathcal{A}_\rho(\tilde{S}_\theta^7)$. Consequently, there is a family of noncommutative principal bundles parameterised by the function r^2 , given by the algebra inclusion $\mathcal{A}_r(S_\theta^4) \hookrightarrow \mathcal{A}_r(S_\theta^7)$. Similarly, there is a family of $\mathrm{SU}(2)$ principal bundles given by the inclusion $\mathcal{A}_\rho(\tilde{S}_\theta^4) \hookrightarrow \mathcal{A}_\rho(\tilde{S}_\theta^7)$. The discussion above shows that the coaction of $\mathcal{B}(\mathrm{SL}_\theta(2, \mathbb{H}))$ serves to map the former family onto the latter. For further details, we refer to [11, 3].

Proposition 5.1. *With $*_\theta : \Omega^2(S_{\theta;r^2}^4) \rightarrow \Omega^2(S_{\theta;r^2}^4)$ the Hodge operator on the sphere $S_{\theta;r^2}^4$ of fixed radius r^2 , the braided Hopf algebra $\mathcal{B}(\mathrm{SL}_\theta(2, \mathbb{H}))$ coacts on $\Omega^2(S_{\theta;r^2}^4)$ by conformal transformations, that is*

$$\Delta_L(*_\theta \omega) = (\mathrm{id} \otimes *_\theta) \Delta_L(\omega) \quad \text{for all } \omega \in \Omega^2(S_{\theta;r^2}^4).$$

Proof. The coaction Δ_L of $\mathcal{B}(\mathrm{SL}_\theta(2, \mathbb{H}))$ is extended to forms $\Omega(S_{\theta;r^2}^4)$ by requiring it to commute with d , namely $\Delta_L(d\omega) = (\mathrm{id} \otimes d) \Delta_L(\omega)$ for all $\omega \in \Omega(S_{\theta;r^2}^4)$. Now the coaction Δ_L is given by the classical action of $\mathrm{SL}(2, \mathbb{H})$ on $\Omega(S_{\theta=0;r^2}^4)$ as vector spaces and only the products on $\mathcal{A}(\mathrm{SL}(2, \mathbb{H}))$ and $\mathcal{A}(S^4)$ are deformed. Since $*_\theta$ coincides with the classical Hodge operator $*$ on $\Omega(S_{\theta;r^2}^4) \simeq \Omega(S_{\theta=0;r^2}^4)$ as vector spaces, the result follows from the classical fact that $\mathrm{SL}(2, \mathbb{H})$ acts on S^4 by conformal transformations. \square

In order to proceed we need to slightly enlarge all of our algebras and assume that the quantity r^2 is invertible with inverse element r^{-2} , and shall henceforth assume that this is done without change of notation. In terms of our families of seven-spheres, this means that we now think of $\mathcal{A}_r(S_\theta^7)$ in the same way as before but without the ‘origin’ in \mathbb{C}_θ^4 , which corresponds to the ‘sphere of radius zero’.

We also define an inverse ρ^{-2} for the quantity ρ^2 , extending the coaction to the new elements by $\rho^{-2} := \Delta_L(r^{-2})$. This gives a well-defined coaction,

$$\Delta_L : \mathcal{A}_r(S_\theta^7) \rightarrow \mathcal{B}(\mathrm{SL}_\theta(2, \mathbb{H})) \underline{\otimes} \mathcal{A}_r(S_\theta^7),$$

for which $\mathcal{A}_r(S_\theta^7)$ is a braided $\mathcal{B}(\mathrm{SL}_\theta(2, \mathbb{H}))$ -comodule algebra. Following the above, for the image under Δ_L we write $\mathcal{A}_\rho(\tilde{S}_\theta^7) := \Delta_L(\mathcal{A}_r(S_\theta^7))$, noting that ρ^2 and ρ^{-2} are central in $\mathcal{A}_\rho(\tilde{S}_\theta^7)$ since r^2 and r^{-2} are central in $\mathcal{A}_r(S_\theta^7)$. In these new terms, the construction of the defining projector of $\mathcal{A}_r(S_\theta^4)$ needs only a minor modification. We now use

$$(5.6) \quad \mathbf{u} := \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ -z_2^* & z_1^* & -z_4^* & z_3^* \end{pmatrix}^t,$$

to denote the same matrix as in Eq. (3.2) but now without imposing the sphere relations, so that now we have $\mathbf{u}^* \mathbf{u} = r^2$. It follows that the matrix

$$(5.7) \quad \mathbf{q} := \mathbf{u} r^{-2} \mathbf{u}^* = \frac{1}{2} r^{-2} \begin{pmatrix} r^2 + x & 0 & \alpha & -\bar{\mu} \beta^* \\ 0 & r^2 + x & \beta & \mu \alpha^* \\ \alpha^* & \beta^* & r^2 - x & 0 \\ -\mu \beta & \bar{\mu} \alpha & 0 & r^2 - x \end{pmatrix}$$

is a projection whose entries generate the $\mathrm{SU}(2)$ -invariant subalgebra of $\mathcal{A}_r(S_\theta^7)$. Moreover, there is a well-defined left coaction of $\mathcal{B}(\mathrm{SL}_\theta(2, \mathbb{H}))$ on the algebra $\mathcal{A}_r(S_\theta^4)$ generated by the entries of this matrix, given by the same formula as in Eq. (5.5). Writing $\tilde{\mathbf{u}} := \Delta_L(\mathbf{u})$, the image of the projector \mathbf{q} under the braided coaction Δ_L is computed to be

$$(5.8) \quad \tilde{\mathbf{q}} := \tilde{\mathbf{u}} \rho^{-2} \tilde{\mathbf{u}}^* = \frac{1}{2} \rho^{-2} \begin{pmatrix} \rho^2 + \tilde{x} & 0 & \tilde{\alpha} & -\bar{\mu} \tilde{\beta}^* \\ 0 & \rho^2 + x & \tilde{\beta} & \mu \tilde{\alpha}^* \\ \tilde{\alpha}^* & \tilde{\beta}^* & \rho^2 - x & 0 \\ -\mu \tilde{\beta} & \bar{\mu} \tilde{\alpha} & 0 & \rho^2 - \tilde{x} \end{pmatrix}$$

and defines an element in the K-theory of the algebra $\mathcal{B}(\mathrm{SL}_\theta(2, \mathbb{H})) \otimes \mathcal{A}_r(S_\theta^4)$. By construction we have that $\tilde{\mathbf{q}} = \Delta_L(\mathbf{q})$, with the entries of $\tilde{\mathbf{q}}$ explicitly given by

$$\tilde{q}_{kl} = \sum_{\alpha, \beta} \rho^{-2} \hat{A}_{k\alpha} (\hat{A}^*)_{\beta l} \otimes (\mathbf{u} \mathbf{u}^*)_{\alpha\beta} F^{-2}(\tau_\beta \tau_l^*, \tau_\alpha \tau_\beta^*),$$

whence we shall think of $\tilde{\mathbf{q}}$ as a ‘braided family’ of projections parameterised by the algebra $\mathcal{B}(\mathrm{SL}_\theta(2, \mathbb{H}))$.

5.3. The cobosonised transformation algebra. As mentioned, when working in the braided category ${}^{H_F} \mathcal{M}$, we have to remember not only that the algebras are twisted, but that the coactions (as braided morphisms in the category) are also twisted since they involve the tensor product structure of the category. This can be computationally rather awkward, so it is useful to remember that the braided left comodules for $\mathcal{B}(\mathrm{SL}_\theta(2, \mathbb{H}))$ (similarly for $\mathcal{B}(\mathrm{Sp}_\theta(2))$) are in one-to-one correspondence with left comodules for its cobosonisation, which is a Hopf algebra in the ‘ordinary’ sense that we now compute.

From §4.2 the left adjoint coaction of H_F on $\mathcal{B}(\mathrm{M}_\theta(2, \mathbb{H}))$ is given by

$$(5.9) \quad \mathcal{B}(\mathrm{M}_\theta(2, \mathbb{H})) \rightarrow H_F \otimes \mathcal{B}(\mathrm{M}_\theta(2, \mathbb{H})), \quad \hat{A}_{ij} \mapsto \tau_i \tau_j^* \otimes \hat{A}_{ij},$$

for $i, j = 1, \dots, 4$. This coaction makes $\mathcal{B}(\mathrm{M}_\theta(2, \mathbb{H}))$ into a coalgebra in the category ${}^{H_F} \mathcal{M}$ of left H_F -comodules, which means that we may construct the associated crossed coproduct coalgebra $\mathcal{B}(\mathrm{M}_\theta(2, \mathbb{H})) \bowtie_{H_F}$ defined in §2.1. As a vector space it is just $\mathcal{B}(\mathrm{M}_\theta(2, \mathbb{H})) \otimes H_F$, with the cross coproduct defined in Eq. (2.8) working out to be

$$(5.10) \quad \Delta(\hat{A}_{ij} \otimes h) = \sum_{\alpha} \hat{A}_{i\alpha} \otimes \tau_\alpha \tau_j^* h \otimes \hat{A}_{\alpha j} \otimes h$$

on group-like elements $h \in H_F$ and extended by linearity.

With the coquasitriangular structure $\mathcal{R}_F = F^{-2}$ on H_F , we have also an H_F -action,

$$(5.11) \quad H_F \times \mathcal{B}(\mathrm{M}_\theta(2, \mathbb{H})) \rightarrow \mathcal{B}(\mathrm{M}_\theta(2, \mathbb{H})), \quad h \triangleright \hat{A}_{ij} = \hat{A}_{ij} F^{-2}(\tau_i \tau_j^*, h),$$

with $i, j = 1, \dots, 4$. Thus, we may canonically view $\mathcal{B}(\mathrm{M}_\theta(2, \mathbb{H}))$ as an object in the category ${}^{H_F}_{H_F} \mathcal{C}$ of crossed H_F -modules. It follows that we may construct the associated cross

product algebra $\mathcal{B}(\mathcal{M}_\theta(2, \mathbb{H})) \bowtie H_F$, which also has $\mathcal{B}(\mathcal{M}_\theta(2, \mathbb{H})) \otimes H_F$ as an underlying vector space. The cross product defined in Eq. (2.7) works out as

$$(\hat{A}_{ij} \otimes g)(\hat{A}_{kl} \otimes h) = \hat{A}_{ij} \cdot (g \triangleright \hat{A}_{kl}) \otimes gh = \hat{A}_{ij} \cdot \hat{A}_{kl} \otimes gh F^{-2}(\tau_k \tau_l^*, g)$$

for group-like elements $g, h \in H_F$. From §2.1, we know that we have constructed the cobosonised bialgebra $\mathcal{B}(\mathcal{M}_\theta(2, \mathbb{H})) \bowtie H_F$.

Remark 5.2. From the general theory of biproducts, $\mathcal{B}(\mathcal{M}_\theta(2, \mathbb{H})) \bowtie H_F$ contains H_F as a sub-Hopf algebra *via* the projection $\pi_H := \epsilon \otimes \text{id}$. The subalgebra $\mathcal{B}(\mathcal{M}_\theta(2, \mathbb{H}))$ is recovered as the algebra of coinvariants under the right coaction $(\text{id} \otimes \pi_H) \circ \Delta$. Moreover, it is interesting to note that the cobosonisation $\mathcal{B}(\mathcal{M}_\theta(2, \mathbb{H})) \bowtie H_F$ contains the transformation bialgebra $\mathcal{A}(\mathcal{M}_\theta(2, \mathbb{H}))$ constructed in [11] (in fact it is isomorphic to the double cross product $\mathcal{A}(\mathcal{M}_\theta(2, \mathbb{H})) \bowtie H_F$, cf. [14, 15]). This is to be expected, since $\mathcal{A}(\mathcal{M}_\theta(2, \mathbb{H}))$ was constructed as the *universal* transformation bialgebra of $\mathcal{A}(\mathbb{C}_\theta^4)$.

Similarly, we may construct the cobosonisations $\mathcal{B}(\text{SL}_\theta(2, \mathbb{H})) \bowtie H_F$ and $\mathcal{B}(\text{Sp}_\theta(2)) \bowtie H_F$ of the braided Hopf algebras $\mathcal{B}(\text{SL}_\theta(2, \mathbb{H}))$ and $\mathcal{B}(\text{Sp}_\theta(2))$. The antipodes on these Hopf algebras are given by Eq. (2.9) and come out on generators to be

$$S(\hat{A}_{ij} \otimes h) = (1 \otimes \tau_j \tau_i^* h^*)(\underline{S}(\hat{A}_{ij}) \otimes 1),$$

with \underline{S} the braided antipode of $\mathcal{B}(\text{SL}_\theta(2, \mathbb{H}))$ and of $\mathcal{B}(\text{Sp}_\theta(2))$.

As a result of this construction, there is a coaction of $\mathcal{B}(\mathcal{M}_\theta(2, \mathbb{H})) \bowtie H_F$ on $\mathcal{A}_r(S_\theta^7)$ and its various subalgebras. Explicitly, we have a coaction

$$(5.12) \quad \mathcal{A}_r(S_\theta^7) \rightarrow (\mathcal{B}(\text{SL}_\theta(2, \mathbb{H})) \bowtie H_F) \otimes \mathcal{A}_r(S_\theta^7), \quad \mathbf{u}_{ia} \mapsto \sum_\beta \hat{A}_{i\beta} \otimes \tau_\beta \otimes \mathbf{u}_{\beta a},$$

on $\mathcal{A}_r(S_\theta^7)$, obtained as the coaction of $\mathcal{B}(\mathcal{M}_\theta(2, \mathbb{H}))$ followed by the coaction of H_F . Just as we did for $\mathcal{B}(\text{SL}_\theta(2, \mathbb{H}))$ in Eq. (5.8), we check that this descends to a well-defined coaction on the family of four-spheres $\mathcal{A}_r(S_\theta^4)$. Indeed, coacting upon the projection \mathbf{q} in this way yields a projection, denoted $\tilde{\mathbf{Q}}$, with entries in the algebra $(\mathcal{B}(\text{SL}_\theta(2, \mathbb{H})) \bowtie H_F) \otimes \mathcal{A}_r(S_\theta^4)$. These entries are given explicitly by

$$(5.13) \quad \tilde{\mathbf{Q}}_{kl} = \sum_{\alpha, \beta} \rho^{-2} \hat{A}_{k\alpha} \cdot (\hat{A}^*)_{\beta l} \otimes \tau_\alpha \tau_\beta^* \otimes (\mathbf{u} \mathbf{u}^*)_{\alpha\beta} F^{-2}(\tau_\beta \tau_l^*, \tau_\alpha \tau_\beta^*).$$

We think of $\tilde{\mathbf{Q}}$ as a noncommutative family of projections parameterised by the algebra $\mathcal{B}(\text{SL}_\theta(2, \mathbb{H})) \bowtie H_F$.

6. NONCOMMUTATIVE FAMILIES OF INSTANTONS

We are now in a position to apply the abstract theory described in previous sections to the subject of interest: the construction of instanton connections on the sphere S_θ^4 . We begin by recalling the theory of anti-self-dual connections on S_θ^4 and what it means for two such connections to be gauge equivalent. We then extend this by discussing the notion of equivalent families of connections over S_θ^4 and, in particular, of families of instantons.

We then provide some examples of families of instanton connections. The first example comes from a basic instanton: by acting upon this with various symmetry groups (namely the torus $H_F = \mathcal{A}(\mathbb{T}^2)$, the braided groups $\mathcal{B}(\text{SL}_\theta(2, \mathbb{H}))$ and $\mathcal{B}(\text{Sp}_\theta(2))$, as well as their cobosonisations $\mathcal{B}(\text{SL}_\theta(2, \mathbb{H})) \bowtie H_F$ and $\mathcal{B}(\text{Sp}_\theta(2)) \bowtie H_F$) we generate a variety of different families and discuss which of them are equivalent.

6.1. Connections and gauge equivalence. Here we briefly recall the notion of gauge equivalence for connections on vector bundles over the four-sphere S_θ^4 , the latter equipped with the differential calculus $(\Omega(S_\theta^4), d)$ defined in §3.3.

Let \mathcal{E} be a finitely-generated projective right $\mathcal{A}(S_\theta^4)$ -module endowed with an $\mathcal{A}(S_\theta^4)$ -valued Hermitian structure $\langle \cdot | \cdot \rangle$; this is assumed to be self-dual, meaning that every right $\mathcal{A}(S_\theta^4)$ -module homomorphism $\phi : \mathcal{E} \rightarrow \mathcal{A}(S_\theta^4)$ is represented by some element $\eta \in \mathcal{E}$ under the assignment $\phi(\cdot) = \langle \eta | \cdot \rangle$. A connection on \mathcal{E} is a linear map $\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{A}(S_\theta^4)} \Omega^1(S_\theta^4)$ satisfying the Leibniz rule

$$\nabla(\xi x) = (\nabla \xi)x + \xi \otimes dx \quad \text{for all } \xi \in \mathcal{E}, x \in \mathcal{A}(S_\theta^4).$$

The connection ∇ is said to be compatible with the Hermitian structure on \mathcal{E} if it obeys

$$\langle \nabla \xi | \eta \rangle + \langle \xi | \nabla \eta \rangle = d\langle \xi | \eta \rangle \quad \text{for all } \xi, \eta \in \mathcal{E}, x \in \mathcal{A}(S_\theta^4).$$

By assumption, \mathcal{E} is a direct summand of a free module, that is $\mathcal{E} = P(\mathbb{C}^N \otimes \mathcal{A}(S_\theta^4))$ for some $P \in \text{End}_{\mathcal{A}(S_\theta^4)}(\mathcal{E})$, $P^2 = P = P^*$, which we use to define the so-called Grassmann connection $\nabla_0 := P \circ d$ on \mathcal{E} . It is straightforward to check that ∇_0 is a compatible connection. Any other compatible connection on \mathcal{E} is of the form $\nabla = \nabla_0 + \alpha$, where α is a skew-adjoint element of $\text{Hom}_{\mathcal{A}(S_\theta^4)}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}(S_\theta^4)} \Omega^1(S_\theta^4))$.

The curvature of ∇ is the $\text{End}_{\mathcal{A}(S_\theta^4)}(\mathcal{E})$ -valued two-form ∇^2 , which in the case of the Grassmann connection ∇_0 is easily computed to be $\nabla_0^2 = P(dP)^2$. More generally, the curvature of $\nabla = \nabla_0 + \alpha$ comes out to be

$$\nabla^2 = P(dP)^2 + P(d\alpha)P + \alpha^2.$$

The curvature ∇^2 is said to be *anti-self-dual* if it satisfies the equation

$$*_\theta \nabla^2 = -\nabla^2,$$

where $*_\theta : \Omega^2(S_\theta^4) \rightarrow \Omega^2(S_\theta^4)$ is the Hodge operator on two-forms; if this is the case we say that the connection ∇ is an *instanton*.

The gauge group of \mathcal{E} is defined to be

$$\mathcal{U}(\mathcal{E}) := \left\{ U \in \text{End}_{\mathcal{A}(S_\theta^4)}(\mathcal{E}) \mid \langle U\xi | U\eta \rangle = \langle \xi | \eta \rangle, \text{ for all } \xi, \eta \in \mathcal{E} \right\}.$$

The gauge group $\mathcal{U}(\mathcal{E})$ acts upon the space of compatible connections by

$$\nabla \mapsto \nabla^U := U \nabla U^*$$

for each compatible connection ∇ and each element U of $\mathcal{U}(\mathcal{E})$. Of course, ∇^U is not a ‘new’ connection, rather it expresses ∇ on the ‘transformed bundle’ $U\mathcal{E}$. Thus we say that a pair of connections ∇_1, ∇_2 on \mathcal{E} are *gauge equivalent* if they are related by such a gauge transformation U . In terms of the decomposition $\nabla = \nabla_0 + \alpha$, one finds $\nabla^U = \nabla_0 + \alpha^U$, where $\alpha^U := U(\nabla_0 U^*) + U\alpha U^*$. The curvature transforms to $(\nabla^U)^2 = U \nabla^2 U^*$, so that if ∇ is an instanton connection then so is ∇^U .

6.2. Families of instantons. Let A be a unital $*$ -algebra over \mathbb{C} . In this section we shall investigate what it means to have a family of connections parameterised by the algebra A and define when two such families are equivalent. We begin with the notion of a family of vector bundles parameterised by an algebra.

Definition 6.1. A family of Hermitian vector bundles over S_θ^4 parameterised by the algebra A is a finitely-generated projective right module \mathcal{E} over the algebra $A \otimes \mathcal{A}(S_\theta^4)$ equipped with an $A \otimes \mathcal{A}(S_\theta^4)$ -valued Hermitian structure $\langle \cdot | \cdot \rangle$. We shall give any such a module *via* a self-adjoint idempotent $P \in M_N(A \otimes \mathcal{A}(S_\theta^4))$, $P^2 = P = P^*$, for which $\mathcal{E} := P(A \otimes \mathcal{A}(S_\theta^4))^N$.

We write $A \otimes \Omega^1(S_\theta^4)$ for the tensor product bimodule over the algebra $A \otimes \mathcal{A}(S_\theta^4)$ and extend the differential d on $\mathcal{A}(S_\theta^4)$ to $A \otimes \mathcal{A}(S_\theta^4)$ as $\text{id} \otimes d$.

Definition 6.2. A family of connections over S_θ^4 parameterised by the algebra A consists of a family of Hermitian vector bundles $\mathcal{E} := P(A \otimes \mathcal{A}(S_\theta^4))^N$ over S_θ^4 , together with a linear map

$$\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes_{A \otimes \mathcal{A}(S_\theta^4)} (A \otimes \Omega^1(S_\theta^4)) \simeq \mathcal{E} \otimes_{\mathcal{A}(S_\theta^4)} \Omega^1(S_\theta^4)$$

obeying the Leibniz rule

$$\nabla(\xi x) = (\nabla \xi)x + \xi \otimes (\text{id} \otimes d)x$$

for all $\xi \in \mathcal{E}$, $x \in A \otimes \mathcal{A}(S_\theta^4)$. The family is said to be *compatible* with the Hermitian structure if it obeys $\langle \nabla \xi | \eta \rangle + \langle \xi | \nabla \eta \rangle = (\text{id} \otimes d)\langle \xi | \eta \rangle$ for all $\xi \in \mathcal{E}$, $x \in A \otimes \mathcal{A}(S_\theta^4)$.

On the family of Hermitian vector bundles $\mathcal{E} = P(A \otimes \mathcal{A}(S_\theta^4))^N$ over S_θ^4 , there is the associated *family of Grassmann connections* $\nabla_0 = P \circ (\text{id} \otimes d)$. More generally, we can always express a family of connections in the form $\nabla = \nabla_0 + \omega$, where ω is an element of $\text{End}_{A \otimes \mathcal{A}(S_\theta^4)}(\mathcal{E}, \mathcal{E} \otimes_{A \otimes \mathcal{A}(S_\theta^4)} (A \otimes \Omega^1(S_\theta^4))) \simeq \text{End}_{A \otimes \mathcal{A}(S_\theta^4)}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}(S_\theta^4)} \Omega^1(S_\theta^4))$.

Definition 6.3. A family of instantons over S_θ^4 is a family of compatible connections ∇ over S_θ^4 whose curvature ∇^2 obeys the anti-self-duality equation

$$(\text{id} \otimes *_\theta) \nabla^2 = -\nabla^2,$$

where $*_\theta$ is the Hodge operator on $\Omega^2(S_\theta^4)$.

We also need to generalise gauge equivalence to incorporate families of connections.

Definition 6.4. Let $\mathcal{E} := P(A \otimes \mathcal{A}(S_\theta^4))^N$ be a family of Hermitian vector bundles parameterised by the algebra A . The *gauge group* of \mathcal{E} is

$$\mathcal{U}(\mathcal{E}) := \{U \in \text{End}_{A \otimes \mathcal{A}(S_\theta^4)}(\mathcal{E}) \mid \langle U\xi | U\eta \rangle = \langle \xi | \eta \rangle \text{ for all } \xi, \eta \in \mathcal{E}\}.$$

We say that two families of compatible connections ∇_1, ∇_2 on \mathcal{E} are *equivalent families* and write $\nabla_1 \sim \nabla_2$ if they are related by the action of the gauge group, *i.e.* there exists $U \in \mathcal{U}(\mathcal{E})$ such that $\nabla_2 = U\nabla_1 U^*$.

Remark 6.5. Where $A = \mathbb{C}$ (*i.e.* for a family parameterised by a one-point space), the above relation reduces to the usual definition of gauge equivalence of connections. In the case where the families ∇_1, ∇_2 are Grassmann families associated to projections $P_1, P_2 \in M_N(A \otimes \mathcal{A}(S_\theta^4))$, equivalence means that $P_2 = UP_1 U^*$ for some unitary U .

Proposition 6.6. Let ∇ be a family of connections over S_θ^4 parameterised by the algebra A . Then for each unital $*$ -algebra morphism $\phi : A \rightarrow B$ there is an induced family $\phi_* \nabla$ of connections parameterised by the algebra B . This operation obeys the functorial properties

$$(\phi_1 \circ \phi_2)_* = \phi_{1*} \circ \phi_{2*}, \quad (\text{id}_A)_* = \text{id},$$

and is compatible with the gauge equivalence \sim in that $\nabla_1 \sim \nabla_2$ implies $\phi_* \nabla_1 \sim \phi_* \nabla_2$.

Proof. Let \mathcal{E}_A be a finitely-generated projective $A \otimes \mathcal{A}(S_\theta^4)$ -module and let ∇ be a connection on \mathcal{E}_A as in Definition 6.2. If $\phi : A \rightarrow B$ is a unital $*$ -algebra map then we define a finitely-generated projective right $B \otimes \mathcal{A}(S_\theta^4)$ -module \mathcal{E}_B by

$$\mathcal{E}_B := \mathcal{E}_A \otimes_{A \otimes \mathcal{A}(S_\theta^4)} (B \otimes \mathcal{A}(S_\theta^4)),$$

where $B \otimes \mathcal{A}(S_\theta^4)$ is thought of as a left $A \otimes \mathcal{A}(S_\theta^4)$ -module *via* the map $\phi \otimes \text{id}_{\mathcal{A}(S_\theta^4)}$. The induced connection $\phi_* \nabla$ is defined by

$$\phi_* \nabla := \nabla \otimes \text{id}_B \otimes \text{id}_{\mathcal{A}(S_\theta^4)} + \text{id}_{\mathcal{E}_A} \otimes \text{id}_B \otimes d$$

with respect to the above decomposition of \mathcal{E}_B . The functorial properties of ϕ_* are obvious; the fact that ϕ_* respects unitary equivalence is also clear. \square

Proposition 6.7. *Let ∇ be a family of connections over S_θ^4 parameterised by the algebra A and let $\phi : A \rightarrow B$ be a morphism of $*$ -algebras. Then the curvature of the induced family $\phi_* \nabla$ is equal to the curvature of ∇ . In particular, if ∇ is a family of instantons then so is $\phi_* \nabla$.*

Proof. The curvature of $\phi_* \nabla$ is computed as follows. For each $\xi \in \mathcal{E}_A$, $b \in B$, $y \in \mathcal{A}(S_\theta^4)$ we have

$$\begin{aligned} (\phi_* \nabla)^2(\xi \otimes b \otimes y) &= (\phi_* \nabla)((\nabla \xi) \otimes b \otimes y + \xi \otimes b \otimes dy) \\ &= (\nabla^2 \xi) \otimes b \otimes y - (\nabla \xi) \otimes b \otimes dy + (\nabla \xi) \otimes b \otimes dy + \xi \otimes b \otimes d^2 y \\ &= (\nabla^2 \xi) \otimes b \otimes y, \end{aligned}$$

where in the second line we have extended $\phi_* \nabla$ using a graded Leibniz rule, as required for it to be well-defined on one-forms. This simply says that, since the curvature ∇^2 is $A \otimes \mathcal{A}(S_\theta^4)$ -linear, it is unaffected when we extend the scalars to $B \otimes \mathcal{A}(S_\theta^4)$. It follows that if ∇^2 is anti-self-dual, then so is $(\phi_* \nabla)^2$. \square

Remark 6.8. Writing **Alg** for the category of unital $*$ -algebras over \mathbb{C} and **Set** for the category of sets, these two propositions say that we have a covariant functor $\mathcal{F} : \mathbf{Alg} \rightarrow \mathbf{Set}$. The functor maps each algebra A to the set $\mathcal{F}(A)$ of equivalence classes of families of instantons parameterised by A . The functor \mathcal{F} is called the *moduli functor*. If we so wish, we may restrict this functor to the sub-category ${}^{H_F} \mathbf{Alg}$ consisting of unital left H_F -comodule $*$ -algebras (*cf.* Appendix A).

Observe that Definition 6.4 only defines an equivalence relation on families parameterised by the same algebra A , whereas Proposition 6.6 provides us with a notion of equivalence for families of instantons which are parameterised by *different* algebras. Indeed, if ∇_1 and ∇_2 are families parameterised by algebras A_1, A_2 , we can think of them as being equivalent if there exists an algebra B and a pair of morphisms $\phi_1 : A_1 \rightarrow B$, $\phi_2 : A_2 \rightarrow B$ such that $\phi_{1*} \nabla_1 \sim \phi_{2*} \nabla_2$.

6.3. The basic instanton. We now turn to the explicit construction of families of connections on S_θ^4 , beginning with a review of the basic instanton constructed in [9]. With \mathbf{q} the projection in Eq. (5.7), we consider the vector bundle and Grassmann connection associated to the complementary projection $\mathbf{p} := 1 - \mathbf{q}$. The Grassmann connection

$$(6.1) \quad \nabla = \mathbf{p} \circ d : \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{A}(S_\theta^4)} \Omega^1(S_\theta^4)$$

has curvature $\nabla^2 = \mathbf{p}(\mathrm{d}\mathbf{p})^2$ which is known to be anti-self-dual,

$$*_\theta(\mathbf{p}(\mathrm{d}\mathbf{p})^2) = -\mathbf{p}(\mathrm{d}\mathbf{p})^2,$$

and is hence an instanton which we call the *basic instanton*. Using noncommutative index theory, the topological charge of the projection \mathbf{p} is computed to be equal to -1 . The reason for going from the projection \mathbf{q} to \mathbf{p} is to obtain a connection with anti-self-dual curvature on a bundle with a fixed rank, just 2 for the case studied. In fact the Grassmann connection defined by \mathbf{q} is known to have self-dual curvature, and would then qualify to be an anti-instanton, given a happy coincidence that is unique to the case of the lowest value of the instanton charge, coming from the fact that the bundles described by the complementary projections \mathbf{q} and \mathbf{p} happen to have the same rank. As we shall see when we come to consider instantons of higher topological charge, the crucial component in this construction is indeed the use of complementary projections to obtain instantons, but the corresponding bundles do not have equal rank.

The simplest way to generate new connections is to act upon the sphere S_θ^4 by a group of symmetries and look at what happens to the basic instanton as a result. The first example of such a symmetry group that we encountered was the two-torus \mathbb{T}^2 , whose action was encoded in §3.2 as a coaction

$$(6.2) \quad \Delta_L : \mathcal{A}(S_\theta^4) \rightarrow H_F \otimes \mathcal{A}(S_\theta^4), \quad \mathbf{q}_{kl} \mapsto \tau_k \tau_l^* \otimes \mathbf{q}_{kl},$$

for $k, l = 1, \dots, 4$. As mentioned above, we are more interested in the complementary projection \mathbf{p} , which transforms in the same way under the coaction of H_F :

$$\Delta_L(\mathbf{p}_{kl}) = \Delta_L(\delta_{kl} - \mathbf{q}_{kl}) = \tau_k \tau_l^* \otimes (\delta_{kl} - \mathbf{q}_{kl}) = \tau_k \tau_l^* \otimes \mathbf{p}_{kl}.$$

This leads immediately to the following fact.

Proposition 6.9. *With Δ_L the coaction of H_F on $\mathcal{A}(S_\theta^4)$ given in (6.2), the projection $\mathbf{p}' := \Delta_L(\mathbf{p})$ defines a family $\nabla' := \mathbf{p}' \circ (\mathrm{id} \otimes d)$ of instantons parameterised by the algebra H_F . The family ∇' is equivalent to the basic instanton $\nabla := \mathbf{p} \circ d$.*

Proof. It is not difficult to check that $\mathbf{p}' := \Delta_L(\mathbf{p})$ is a projection with entries in the algebra $H_F \otimes \mathcal{A}(S_\theta^4)$. We may view \mathbf{p} as a projection in $M_4(H_F \otimes \mathcal{A}(S_\theta^4))$ using the algebra map $\mathcal{A}(S_\theta^4) \hookrightarrow H_F \otimes \mathcal{A}(S_\theta^4)$ defined by $a \mapsto 1 \otimes a$ for each $a \in \mathcal{A}(S_\theta^4)$. Taking U to be the unitary matrix

$$(6.3) \quad U := \mathrm{diag}(\tau_1 \otimes 1, \tau_2 \otimes 1, \tau_3 \otimes 1, \tau_4 \otimes 1) \in M_4(H_F \otimes \mathcal{A}(S_\theta^4)),$$

with τ_j the generators of H_F , it is straightforward to verify that $\mathbf{p}' = U(1 \otimes \mathbf{p})U^*$ and so the two families are equivalent. It follows immediately that the family ∇' also has anti-self-dual curvature. \square

We immediately see how to generate other families of instantons which are equivalent to the basic one. We are not limited to conjugating \mathfrak{p} simply by generators of H_F as in Eq. (6.3): more generally we can take any quadruple (u_1, u_2, u_3, u_4) of unitary elements in H_F with $u_1 = u_2^*$, $u_3 = u_4^*$ and set

$$U = \text{diag } (u_1 \otimes 1, u_2 \otimes 1, u_3 \otimes 1, u_4 \otimes 1) \in M_4(H_F \otimes \mathcal{A}(S_\theta^4)).$$

The resulting projection $U(1 \otimes \mathfrak{p})U^*$ is by definition equivalent to \mathfrak{p} . We still get a family of instantons parameterised by H_F , although it is a different parameterisation from the one in Proposition 6.9. Since the topological charge of the projection \mathfrak{p} is invariant under unitary equivalence, these families also have charge equal to -1 .

6.4. A noncommutative family of instantons. Next we examine the effect of the coaction $\Delta_L : \mathcal{A}_r(S_\theta^4) \rightarrow (\mathcal{B}(\text{SL}_\theta(2, \mathbb{H})) \rtimes H_F) \otimes \mathcal{A}_r(S_\theta^4)$ of the cobosonised transformation algebra on the basic instanton, once again stressing that for a well-defined coaction we have to work not with $\mathcal{A}(S_\theta^4)$ but with the entire family of spheres $\mathcal{A}_r(S_\theta^4)$. Recall from Eq. (5.12) that the coaction is given by

$$\Delta_L : \mathcal{A}_r(S_\theta^4) \rightarrow (\mathcal{B}(\text{SL}_\theta(2, \mathbb{H})) \rtimes H_F) \otimes \mathcal{A}_r(S_\theta^4)$$

on $\mathcal{A}_r(S_\theta^4)$ and hence on the projection \mathfrak{q} , yielding the family of projections $\tilde{\mathfrak{Q}}$ described in Eq. (5.13):

$$(6.4) \quad \tilde{\mathfrak{Q}}_{kl} = \sum_{\alpha, \beta} \rho^{-2} \hat{A}_{k\alpha} \cdot (\hat{A}^*)_{\beta l} \otimes \tau_\alpha \tau_\beta^* \otimes (\mathbf{u}\mathbf{u}^*)_{\alpha\beta} F^{-2}(\tau_\beta \tau_l^*, \tau_\alpha \tau_\beta^*).$$

There is a similar coaction of the same Hopf algebra on the projection \mathfrak{p} , which may be expressed by writing $\mathfrak{p} = 1 - \mathfrak{q}$ and computing

$$\Delta_L(\mathfrak{p}_{kl}) = \Delta_L(\delta_{kl} - \mathfrak{q}_{kl}) = 1 \otimes \delta_{kl} - \tilde{\mathfrak{Q}}_{kl}.$$

We denote the resulting projection by $\tilde{\mathfrak{P}} := \Delta_L(\mathfrak{p})$. Extending the exterior derivative to $(\mathcal{B}(\text{SL}_\theta(2, \mathbb{H})) \rtimes H_F) \otimes \mathcal{A}_r(S_\theta^4)$ as $\text{id} \otimes d$, we get the following result.

Proposition 6.10. *The family $\tilde{\nabla}$ of Grassmann connections defined by $\tilde{\nabla} := \tilde{\mathfrak{P}} \circ (\text{id} \otimes d)$ has anti-self-dual curvature, that is*

$$(\text{id} \otimes *_\theta) \tilde{\mathfrak{P}}((\text{id} \otimes d) \tilde{\mathfrak{P}})^2 = -\tilde{\mathfrak{P}}((\text{id} \otimes d) \tilde{\mathfrak{P}})^2.$$

Proof. By definition, the coaction of $\mathcal{B}(\text{SL}_\theta(2, \mathbb{H})) \rtimes H_F$ on $\mathcal{A}_r(S_\theta^4)$ is given by first coacting with $\mathcal{B}(\text{SL}_\theta(2, \mathbb{H}))$ followed by coacting with H_F . From Proposition 5.1, the braided group $\mathcal{B}(\text{SL}_\theta(2, \mathbb{H}))$ coacts by conformal transformations and so commutes with the Hodge structure $*_\theta$, hence preserving the anti-self-duality. By Proposition 6.9, the Hopf algebra H_F coacts unitarily and hence preserves the curvature, and we know that the $\mathcal{B}(\text{SL}_\theta(2, \mathbb{H}))$ -coaction intertwines the H_F -coaction. Putting these coactions together gives a family of instantons parameterised by the algebra $\mathcal{B}(\text{SL}_\theta(2, \mathbb{H})) \rtimes H_F$. \square

Next we wish to show that the family $\tilde{\nabla}$ has topological charge equal to -1 . Recall that a pair of projections P, Q are said to be Murray-von Neumann equivalent if there exists a partial isometry V such that $P = VV^*$ and $Q = V^*V$.

Lemma 6.11. *The projections $1 \otimes \mathfrak{p}$ and $\tilde{\mathfrak{P}}$ are Murray-von Neumann equivalent in the algebra $M_4((\mathcal{B}(\text{SL}_\theta(2, \mathbb{H})) \rtimes H_F) \otimes \mathcal{A}_r(S_\theta^4))$ and hence have the same topological charge.*

Proof. First one shows as in [11] that the projections $1 \otimes \mathbf{q}$ and $\tilde{\mathbf{Q}}$ are Murray-von Neumann equivalent in the algebra $M_4((\mathcal{B}(\mathrm{SL}_\theta(2, \mathbb{H})) \rtimes_{H_F}) \otimes \mathcal{A}_r(S_\theta^4))$. Indeed, defining a partial isometry $V = (V_{kl}) \in M_4((\mathcal{B}(\mathrm{SL}_\theta(2, \mathbb{H})) \rtimes_{H_F}) \otimes \mathcal{A}_r(S_\theta^4))$ by

$$V_{kl} := \sum_{\alpha} \rho^{-1}(\hat{A}_{k\alpha} \otimes \tau_{\alpha}) \otimes \mathbf{q}_{\alpha l},$$

a straightforward computation shows that $V^*V = 1 \otimes \mathbf{q}$ and $VV^* = \tilde{\mathbf{Q}}$. Since $1 \otimes \mathbf{q}$ and $1 \otimes \mathbf{p}$ are complementary projections, as are $\tilde{\mathbf{Q}}$ and $\tilde{\mathbf{P}}$, it immediately follows that $1 \otimes \mathbf{p}$ and $\tilde{\mathbf{P}}$ are Murray-von Neumann equivalent. One may also show in the same way as in [11] that the topological charge of the projection $\tilde{\mathbf{Q}}$ is equal to 1, whence it follows that $\tilde{\mathbf{P}}$ has topological charge equal to -1 . \square

As we did for the basic instanton, we can produce other families by conjugating $\tilde{\mathbf{P}}$ with a unitary matrix. In particular, we can take any quadruple (u_1, u_2, u_3, u_4) of unitary elements in H_F with $u_1 = u_2^*$, $u_3 = u_4^*$ and define

$$U = \mathrm{diag}(u_1, u_2, u_3, u_4) \in M_4((\mathcal{B}(\mathrm{SL}_\theta(2, \mathbb{H})) \rtimes_{H_F}) \otimes \mathcal{A}_r(S_\theta^4))$$

(we have suppressed the trivial factors in the tensor product in the entries of U). In this case, the conjugated projection $U\tilde{\mathbf{Q}}U^*$ has entries

$$(6.5) \quad \begin{aligned} (U\tilde{\mathbf{Q}}U^*)_{kl} &= U_{k\alpha} \tilde{\mathbf{P}}_{\alpha\beta} (U_{l\beta})^* \\ &= \sum_{\alpha, \beta} \rho^{-2} \hat{A}_{k\alpha} \cdot (\hat{A}^*)_{\beta l} \otimes u_k \tau_{\alpha} \tau_{\beta}^* u_l^* \otimes (\mathbf{u} \mathbf{u}^*)_{\alpha\beta} F^{-2}(\tau_{\beta} \tau_l^*, \tau_{\alpha} \tau_{\beta}^*), \end{aligned}$$

which are elements in the algebra $(\mathcal{B}(\mathrm{SL}_\theta(2, \mathbb{H})) \rtimes_{H_F}) \otimes \mathcal{A}_r(S_\theta^4)$. The conjugated projection $U\tilde{\mathbf{P}}U^*$ yields a family of instantons parameterised by the algebra $\mathcal{B}(\mathrm{SL}_\theta(2, \mathbb{H})) \rtimes_{H_F}$ which is gauge equivalent to the Grassmann family defined by $\tilde{\mathbf{P}}$.

Finally in this section, we also consider the coaction of the Hopf algebra $\mathcal{B}(\mathrm{Sp}_\theta(2)) \rtimes_{H_F}$ on the basic instanton defined by \mathbf{p} , with the following result. We denote by $\tilde{\mathbf{P}}_0$ and $\tilde{\mathbf{Q}}_0$ the images of the projections \mathbf{p} and \mathbf{q} under the coaction $\mathcal{A}(S_\theta^4) \rightarrow (\mathcal{B}(\mathrm{Sp}_\theta(2)) \rtimes_{H_F}) \otimes \mathcal{A}(S_\theta^4)$.

Proposition 6.12. *The Grassmann connection $\tilde{\nabla}_0 := \tilde{\mathbf{P}}_0 \circ (\mathrm{id} \otimes \mathrm{d})$ is a family of instantons parameterised by the Hopf algebra $\mathcal{B}(\mathrm{Sp}_\theta(2)) \rtimes_{H_F}$; the family $\tilde{\nabla}_0$ is equivalent to the basic instanton $\nabla := \mathbf{p} \circ \mathrm{d}$.*

Proof. The fact that $\tilde{\nabla}_0$ is a family of instantons follows in the same way as the proof of Proposition 6.10. We have to show that the projection $\tilde{\mathbf{P}}_0$ is unitarily equivalent to $1 \otimes \mathbf{p}$ in the matrix algebra $M_4((\mathcal{B}(\mathrm{Sp}_\theta(2)) \rtimes_{H_F}) \otimes \mathcal{A}(S_\theta^4))$. The unitary matrix which achieves this is

$$U = (U_{kl}) \in M_4((\mathcal{B}(\mathrm{Sp}_\theta(2)) \rtimes_{H_F}) \otimes \mathcal{A}(S_\theta^4)), \quad U_{kl} := (\hat{A}_{kl} \otimes \tau_l) \otimes 1,$$

where the elements \hat{A}_{kl} in this case denote the generators of $\mathcal{B}(\mathrm{Sp}_\theta(2))$. It is a straightforward computation to check that $\tilde{\mathbf{Q}}_0 = U(1 \otimes \mathbf{q})U^*$ and hence that $\tilde{\mathbf{P}}_0 = U(1 \otimes \mathbf{p})U^*$, whence the result. In these computations it is important to note that, thanks to the $*$ -structure on $\mathcal{B}(\mathrm{Sp}_\theta(2)) \rtimes_{H_F}$, the matrix U^* has entries $(U^*)_{kl} = F^{-2}(\tau_k \tau_l^*, \tau_k^*) (\hat{A}^*)_{kl} \otimes \tau_k^* \otimes 1$. \square

In summary, we have shown how to construct various families of instantons on S_θ^4 all having topological charge equal to -1 . Clearly, the parameter spaces for these families are not necessarily ‘optimal’, in the sense that some of the parameters may correspond

to gauge equivalent instantons. It is of central interest and of course only natural to investigate how to remove these extra gauge parameters and leave parameter spaces which describe gauge equivalence classes of instantons. This is addressed in the next section.

7. PARAMETER SPACES FOR CHARGE ONE INSTANTONS

In the classical case, if a Lie group G acts (freely, let us say) on a smooth manifold P , then one can consider the corresponding space of orbits P/G . In cases when P is a parameter space for a family of instantons such that the action of G preserves gauge equivalence classes, then one can always construct a new family of instantons labeled by the more “efficient” parameter space P/G (the latter clearly having less redundancy).

In the noncommutative setting, where we allow for noncommutative parameter spaces, our strategy is analogous, with group actions and spaces being replaced by coactions of Hopf algebras and ‘noncommutative quotients’. We show that, in the situation where the coaction of a Hopf algebra on a parameter space results in a gauge equivalent family of instantons, there is a family of instantons parameterised by the noncommutative quotient space (the algebra of coinvariants for the coaction).

As mentioned in Remark 6.8, we think of the moduli functor as a functor whose source is the category ${}^{H_F}\mathbf{Alg}$ of unital left H_F -comodule $*$ -algebras, *i.e.* we consider parameter spaces described by algebras which carry a left H_F -coaction. This is in keeping with our strategy of viewing the passage from classical to quantum as a ‘quantisation functor’ as in §2.2. In particular, this means that all quantum groups we consider are braided Hopf algebras in the category, and all coactions are required to be morphisms in the category and hence braided as well.

7.1. Removing the $\mathcal{B}(\mathrm{Sp}_\theta(2))$ gauge parameters. For the sake of brevity, in this section we write $A := \mathcal{B}(\mathrm{SL}_\theta(2, \mathbb{H})) \rtimes H_F$ and $C := \mathcal{B}(\mathrm{Sp}_\theta(2))$. Of all the families of charge one instantons that we have constructed, the largest is the one parameterised by the noncommutative algebra A , and we seek to make it smaller by quotienting away the parameters corresponding to gauge equivalence. In this section, we consider a coaction of C on the parameter space A and show that it generates a gauge equivalent family of instantons. These gauge parameters are removed by constructing the corresponding quantum quotient of A by C .

Recall that the braided Hopf algebra $\mathcal{B}(\mathrm{Sp}_\theta(2))$ is the quotient of $\mathcal{B}(\mathrm{SL}_\theta(2, \mathbb{H}))$ by an ideal \mathcal{I}_θ , obtained as a twist of the ideal \mathcal{I} defined in Eq. (4.3). Let us write

$$\Pi_\theta : \mathcal{B}(\mathrm{SL}_\theta(2, \mathbb{H})) \rightarrow \mathcal{B}(\mathrm{Sp}_\theta(2))$$

for the canonical projection. Using this we can define a braided left coaction of C on A as follows. Note that A is canonically an object in the category ${}^{H_F}\mathcal{M}$ via the tensor product H_F -coaction, whence we may form the braided tensor product algebra $C \underline{\otimes} A$. We use the notation $\Delta_F(a) = a_{(1)} \otimes a_{(2)}$ for the braided coproduct Δ_F of $\mathcal{B}(\mathrm{SL}_\theta(2, \mathbb{H}))$.

Lemma 7.1. *There is a braided left coaction of the braided Hopf algebra C on the parameter space A defined by the formula*

$$\delta_L : A \rightarrow C \underline{\otimes} A, \quad \delta_L(a \otimes h) = \Pi_\theta(a_{(1)}) \otimes a_{(2)} \otimes h,$$

for which A is a braided left C -comodule algebra.

Proof. The fact that δ_L defines a braided coaction is immediate from the fact that both the coproduct Δ_F and the projection Π_θ are morphisms in the category ${}^{H_F}\mathcal{M}$, hence so is the composition $\delta_L = \Pi_\theta \circ \Delta_F$. To check that δ_L respects the algebra structure of A , we compute on generators that

$$\begin{aligned}
\delta_L(\hat{A}_{ij} \otimes h) \cdot \delta_L(\hat{A}_{kl} \otimes g) &= \sum_{\alpha, \beta} (\Pi_\theta(\hat{A}_{i\alpha}) \otimes \hat{A}_{\alpha j} \otimes h) \cdot (\Pi_\theta(\hat{A}_{k\beta}) \otimes \hat{A}_{\beta l} \otimes g) \\
&= \sum_{\alpha, \beta} \Pi_\theta(\hat{A}_{i\alpha} \cdot \hat{A}_{k\beta}) \otimes \hat{A}_{\alpha j} \cdot \hat{A}_{\beta l} \otimes hg \times \\
&\quad \times F^{-2}(\tau_k \tau_\beta^*, \tau_\alpha \tau_j^* h) F^{-2}(\tau_\beta \tau_l^*, h) \\
&= \sum_{\alpha, \beta} \Pi_\theta(\hat{A}_{i\alpha} \cdot \hat{A}_{k\beta}) \otimes \hat{A}_{\alpha j} \cdot \hat{A}_{\beta l} \otimes hg \times \\
&\quad \times F^{-2}(\tau_k \tau_l^*, h) F^{-2}(\tau_k \tau_\beta^*, \tau_\alpha \tau_j^*) \\
&= \delta_L(\hat{A}_{ij} \cdot \hat{A}_{kl} \otimes hg) F^{-2}(\tau_k \tau_l^*, h) \\
&= \delta_L((\hat{A}_{ij} \otimes h) \cdot (\hat{A}_{kl} \otimes g)),
\end{aligned}$$

as required for a braided comodule algebra. \square

This establishes the quantum analogue of a group action on our parameter space. The following lemma tells us that this action is by gauge transformations. Since our strategy is to compare pairs of parameter spaces by looking to see if they define unitarily equivalent families of instantons (through an application of the moduli functor defined in Remark 6.8), the correct way to interpret the effect of the coaction δ_L on the family defined by the parameter space A is by coacting upon the entries of the projection $\mathbf{P} \in M_4(A \otimes \mathcal{A}_r(S_\theta^4))$ by $\delta_L \otimes \text{id}$, that is to say by leaving the algebra $\mathcal{A}_r(S_\theta^4)$ alone in this coaction.

Proposition 7.2. *The image $\delta_L(\tilde{\mathbf{P}})$ of the projection $\tilde{\mathbf{P}}$ under the coaction $\delta_L \otimes \text{id}$ is unitarily equivalent to the projection $1 \otimes \tilde{\mathbf{P}}$ in the algebra $M_4((C \otimes A) \otimes \mathcal{A}_r(S_\theta^4))$.*

Proof. We first consider the effect of the coaction $\delta_L \otimes \text{id}$ on the projection $\tilde{\mathbf{Q}}$ in (6.4):

$$\begin{aligned}
(\delta_L \otimes \text{id})(\tilde{\mathbf{Q}}_{kl}) &= \sum_{\alpha, \beta, \gamma, \delta} (1 \otimes \rho^{-2}) \Pi_\theta(\hat{A}_{k\gamma} \cdot (\hat{A}^*)_{\delta l}) \otimes \hat{A}_{\gamma\alpha} \cdot (\hat{A}^*)_{\beta\delta} \otimes \tau_\alpha \tau_\beta^* \otimes (\mathbf{u}\mathbf{u}^*)_{\alpha\beta} \times \\
&\quad \times F^{-2}(\tau_\beta \tau_l^*, \tau_\beta^*) F^{-2}(\tau_\delta \tau_l^*, \tau_\beta \tau_\delta^*) F^{-2}(\tau_\delta \tau_l^*, \tau_\gamma) F^{-2}(\tau_\beta \tau_\delta^*, \tau_\alpha).
\end{aligned}$$

It is a straightforward calculation, along the same lines as Proposition 6.12, to check that the same effect is achieved by conjugation with the unitary matrix

$$U = (U_{kl}) \in M_4((C \otimes A) \otimes \mathcal{A}_r(S_\theta^4)), \quad U_{kl} = (\Pi_\theta(\hat{A}_{kl}) \otimes 1) \otimes 1,$$

i.e. we have $\delta_L(\tilde{\mathbf{Q}}) = U(1 \otimes \tilde{\mathbf{Q}})U^*$. From the fact that $\tilde{\mathbf{P}}$ is complementary to $\tilde{\mathbf{Q}}$ it follows immediately that $\delta_L(\tilde{\mathbf{P}}) = U(1 \otimes \tilde{\mathbf{P}})U^*$, as required. \square

The subalgebra $C = \mathcal{B}(\text{Sp}_\theta(2))$ of $A = \mathcal{B}(\text{SL}_\theta(2, \mathbb{H})) \rtimes H_F$ thus consists entirely of gauge parameters which we would like to remove. This reduction of parameters is performed by computing the quantum quotient of A by the coaction of C .

Proposition 7.3. *The algebra of coinvariants for the coaction δ_L ,*

$$\{a \in \mathcal{B}(\text{SL}_\theta(2, \mathbb{H})) \rtimes H_F \mid \delta_L(a) = 1 \otimes a\},$$

is isomorphic to the subalgebra $\mathcal{B}(\mathbf{M}_\theta) \rtimes_{H_F}$ of $\mathcal{B}(\mathrm{SL}_\theta(2, \mathbb{H})) \rtimes_{H_F}$, where $\mathcal{B}(\mathbf{M}_\theta)$ is the subalgebra of $\mathcal{B}(\mathrm{SL}_\theta(2, \mathbb{H}))$ generated by the elements

$$m_{ij} := \sum_{\alpha} (\hat{A}_{\alpha i})^* \cdot \hat{A}_{\alpha j}, \quad i, j = 1, \dots, 4.$$

Proof. Since the relations in $\mathcal{B}(\mathrm{Sp}_\theta(2))$ are quadratic in the generators \hat{A}_{kl} and their conjugates, the generators of the algebra of coinvariants must be at least quadratic in them. The key relations here are those coming from the antipode, namely

$$\sum_{\alpha} (A^*)_{i\alpha} A_{\alpha j} = \delta_{ij}, \quad i, j = 1, \dots, 4.$$

In order for the first leg of the tensor product in $\delta_L(a \otimes h)$ to involve these relations, we have to take $a = \sum_{\alpha} (\hat{A}_{\alpha i})^* \cdot \hat{A}_{\alpha j}$. Moreover, we compute that for all group-like elements $h \in H_F$ we have

$$\begin{aligned} \delta_L \left(\sum_{\alpha} (\hat{A}_{\alpha i})^* \cdot \hat{A}_{\alpha j} \otimes h \right) &= \sum_{\alpha, \beta, \gamma} \Pi_{\theta} \left((\hat{A}_{\alpha \beta})^* \cdot \hat{A}_{\alpha \gamma} \right) \otimes (\hat{A}_{\beta i})^* \cdot \hat{A}_{\gamma j} \otimes h \\ &= \sum_{\beta, \gamma} \delta_{\beta \gamma} \otimes (\hat{A}_{\beta i})^* \cdot \hat{A}_{\gamma j} \otimes h \\ &= 1 \otimes \sum_{\beta} ((\hat{A}_{\beta i})^* \cdot \hat{A}_{\beta j}) \otimes h. \end{aligned}$$

The identification of the algebra of coinvariants as $\mathcal{B}(\mathbf{M}_\theta) \rtimes_{H_F}$ is now obvious. \square

This gives us a new parameter space which we denote by $B := \mathcal{B}(\mathbf{M}_\theta) \rtimes_{H_F}$. Next we have to check that it really does parameterise a family of instantons. To this end, let \mathcal{E}_A denote the finitely-generated projective $A \otimes \mathcal{A}_r(S_\theta^4)$ -module defined by the projection \tilde{P} . We define

$$\mathcal{E}_B = \mathcal{E}_A^{\mathrm{co}C} := \{ \xi \in \mathcal{E}_A \mid \delta_L(\xi) = 1 \otimes \xi \}$$

to be the vector space of coinvariant elements in \mathcal{E}_A . It is clear that the right $A \otimes \mathcal{A}_r(S_\theta^4)$ -module structure on \mathcal{E}_A survives as a right $B \otimes \mathcal{A}_r(S_\theta^4)$ -module structure on \mathcal{E}_B .

Lemma 7.4. *The induced module*

$$\mathcal{E}_B \otimes_{B \otimes \mathcal{A}_r(S_\theta^4)} (A \otimes \mathcal{A}_r(S_\theta^4)) \simeq \mathcal{E}_B \otimes_B A$$

is canonically isomorphic to \mathcal{E}_A as a right $A \otimes \mathcal{A}_r(S_\theta^4)$ -module.

Proof. The proof of this assertion goes along the lines of [20], noting that the proof there is given in terms of ‘ordinary’ rather than braided coactions. However, the proof goes through in the braided case as well: in fact one may view the inclusion $B \hookrightarrow A$ purely as an extension of coalgebras and still make the same conclusions [5], so that the (braided or ordinary) algebra structure of the extension is not important. The strategy is to check that the canonical algebra inclusion $\iota : B \hookrightarrow A$ is a faithfully flat (braided) C -Hopf-Galois extension: this follows from the facts that the canonical linear map

$$(7.1) \quad \chi : A \otimes_B A \rightarrow C \otimes A, \quad a \otimes a' \mapsto \delta_L(a) a'$$

is a bijection (as is always the case for coactions defined in this way by a Hopf algebra projection) and that $C = \mathcal{B}(\mathrm{Sp}_\theta(2))$ is a cosemisimple Hopf algebra. From this, it follows that the category ${}^C\mathcal{M}_{A \otimes \mathcal{A}_r(S_\theta^4)}$ of left C -comodule right $A \otimes \mathcal{A}_r(S_\theta^4)$ -modules is equivalent to the category $\mathcal{M}_{B \otimes \mathcal{A}_r(S_\theta^4)}$ of right $B \otimes \mathcal{A}_r(S_\theta^4)$ -modules (each viewed as a sub-category of ${}^{H_F}\mathcal{M}$). \square

The module \mathcal{E}_B thus defines a family of Hermitian vector bundles over S_θ^4 parameterised by the algebra B . On the projective $A \otimes \mathcal{A}_r(S_\theta^4)$ -module $\mathcal{E}_A = \tilde{\mathbf{P}}(A \otimes \mathcal{A}_r(S_\theta^4))^4$ we have the family of instantons, $\nabla_A := \tilde{\mathbf{P}} \circ (\text{id} \otimes d)$, as constructed in §6.4. The next proposition gives us the required family of instanton connections on the family of bundles \mathcal{E}_B .

Proposition 7.5. *Let $\iota : B \hookrightarrow A$ be the canonical algebra inclusion. Then there exists a Grassmann family ∇_B of instantons parameterised by the algebra B , unique up to unitary equivalence, with the property that $\iota_*(\nabla_B) = \nabla_A$.*

Proof. Recall that ∇_A is the Grassmann connection on the projective $A \otimes \mathcal{A}_r(S_\theta^4)$ -module $\mathcal{E}_A = \tilde{\mathbf{P}}(A \otimes \mathcal{A}_r(S_\theta^4))^4$. From the above discussion, the coinvariant sub-module $\mathcal{E}_B := \mathcal{E}_A^{\text{co}C}$ is finitely-generated and projective as a right $B \otimes \mathcal{A}_r(S_\theta^4)$ -module, and hence defined by a projection \mathbf{P}_B , unique up to unitary equivalence. We define a Grassmann family of connections on \mathcal{E}_B by $\nabla_B := \mathbf{P}_B \circ (\text{id} \otimes d)$. Since the induced module $\mathcal{E}_B \otimes_B A$ is canonically isomorphic to \mathcal{E}_A , the induced family of connections $\iota_*(\nabla_B)$ defined in Proposition 6.6 must be the same as the family ∇_A (up to equivalence). From the proof of Proposition 6.7, we see that the curvature of ∇_B is the same as the curvature of ∇_A , which means that the curvature of the family ∇_B must also be anti-self-dual. \square

7.2. Removing the H_F gauge parameters. We have thus removed the gauge parameters corresponding to the braided group $\mathcal{B}(\text{Sp}_\theta(2))$, yielding a family of instantons parameterised by the algebra B . The next step is to quotient away the parameters corresponding to the algebra H_F .

Our strategy is the same as before: we remove these parameters by considering a (braided) coaction of H_F on B . By showing that this coaction is by unitary gauge transformations, we then pass to the parameter space described by the quantum quotient of B by H_F . This is a very delicate process, however, since there are many different ways in which H_F can coact upon the parameter space $B = \mathcal{B}(\mathbf{M}_\theta) \bowtie H_F$, whence there are many different ways in which we can make the quotient.

Once again we consider all of our parameter spaces as being objects in the category ^{H_F}Alg , in particular noting that H_F is canonically an object in the category *via* the left regular coaction. This means that we can form the braided tensor product algebras $H_F \underline{\otimes} A$ and $H_F \underline{\otimes} B$ as objects in the category.

Lemma 7.6. *Let $u := (u_1, u_2, u_3, u_4)$ be unitary elements of H_F such that $u_1^* = u_2$, $u_3^* = u_4$. Then there is a left coaction $\delta_u : A \rightarrow H_F \underline{\otimes} A$ defined on by*

$$\delta_u(\hat{A}_{kl} \otimes h) = u_k u_l^* h \otimes \hat{A}_{kl} \otimes h, \quad k, l = 1, \dots, 4$$

for each group-like element $h \in H_F$ and extended as a braided $$ -algebra map.*

Proof. We check the conditions for δ_u to define a braided coaction of H_F :

$$\begin{aligned} ((\text{id} \otimes \delta_u) \circ \delta_u)(\hat{A}_{kl} \otimes h) &= (\text{id} \otimes \delta_u)(u_k u_l^* h \otimes \hat{A}_{kl} \otimes h) \\ &= u_k u_l^* h \otimes u_k u_l^* h \otimes \hat{A}_{kl} \otimes h \\ &= ((\Delta \otimes \text{id}) \circ \delta_u)(\hat{A}_{kl} \otimes h); \\ ((\epsilon \otimes \text{id}) \circ \delta_u)(\hat{A}_{kl} \otimes h) &= \epsilon(u_k u_l^* h) \hat{A}_{kl} \otimes h = \hat{A}_{kl} \otimes h, \end{aligned}$$

where in the last line we have used the fact that $\epsilon(h) = 1$ for all group-like elements $h \in H_F$. We then extend δ_u as a braided $*$ -algebra map to obtain the result. \square

As before, we extend the coaction $\delta_u : A \rightarrow H_F \underline{\otimes} A$ to a coaction on the algebra $A \otimes \mathcal{A}(S_\theta^4)$ by $\delta_u \otimes \text{id}$. In this way we can coact upon the projection $\tilde{\mathbf{P}}$ with H_F . Our next result is that this coaction is by gauge transformations.

Proposition 7.7. *Let $u := (u_1, u_2, u_3, u_4)$ be unitary elements of H_F as above. Then the image $(\delta_u \otimes \text{id})(\tilde{\mathbf{P}})$ of the projection $\tilde{\mathbf{P}}$ under the coaction $\delta_u \otimes \text{id}$ is unitarily equivalent to the projection $1 \otimes \tilde{\mathbf{P}}$ in the algebra $M_4((H_F \underline{\otimes} A) \otimes \mathcal{A}_r(S_\theta^4))$.*

Proof. We first consider the effect of the coaction δ_u on the projection $\tilde{\mathbf{Q}}$:

$$\begin{aligned} (\delta_u \otimes \text{id})(\tilde{\mathbf{Q}}_{kl}) &= (1 \otimes \rho^{-2}) \sum_{\alpha, \beta} (u_k u_\alpha^* u_\beta u_l^* \tau_\alpha \tau_\beta^*) \otimes \hat{A}_{k\alpha} \cdot (\hat{A}^*)_{\beta l} \otimes \tau_\alpha \tau_\beta^* \otimes (uu^*)_{\alpha\beta} \times \\ &\quad \times F^{-2}(\tau_\beta \tau_l^*, \tau_\alpha \tau_\beta^*). \end{aligned}$$

It is a straightforward calculation to check that the same effect can be achieved by conjugating with the unitary diagonal matrix

$$U = (U_{kl}) \in M_4((H_F \underline{\otimes} A) \otimes \mathcal{A}_r(S_\theta^4)), \quad U_{kl} = \text{diag}((u_k \tau_k \otimes 1) \otimes 1 \mid k = 1, \dots, 4),$$

i.e. we have $(\delta_u \otimes \text{id})(\tilde{\mathbf{Q}}) = U(1 \otimes \tilde{\mathbf{Q}})U^*$. From the fact that $\tilde{\mathbf{P}}$ is complementary to $\tilde{\mathbf{Q}}$ it follows immediately that $(\delta_u \otimes \text{id})(\tilde{\mathbf{P}}) = U(1 \otimes \tilde{\mathbf{P}})U^*$, as required. \square

Each of the coactions δ_u therefore gives us an equally valid way of quotienting the parameter space and removing gauge freedom. It is clear that the coaction δ_u descends to a coaction on the algebra $B = \mathcal{B}(M_\theta) \rtimes H_F$, whence we have the following result.

Proposition 7.8. *Let $u := (u_1, u_2, u_3, u_4)$ be unitary elements of H_F as above. Then the algebra of coinvariants*

$$\mathcal{A}(M_\theta^u) := \{b \in B \mid \delta_u(b) = 1 \otimes b\}$$

for the coaction δ_u is generated by the elements $M_{ij}^u := \mathbf{m}_{ij} \otimes u_i^ u_j$ for $i, j = 1, \dots, 4$.*

Proof. This is a matter of noticing that on generators $\mathbf{m}_{ij} \otimes h$ of B the coaction δ_u has the form

$$\delta_u(\mathbf{m}_{ij} \otimes h) = u_i u_j^* h \otimes \mathbf{m}_{ij} \otimes h.$$

This implies that the algebra of coinvariants is generated by elements for which $h = u_i^* u_j$, as claimed. \square

By removing gauge parameters, we have thus produced a more efficient parameter space $M := \mathcal{A}(M_\theta^u)$. Again we have to check that M really does parameterise a family of instantons. This time we define

$$\mathcal{E}_M = \mathcal{E}_B^{\text{co } H_F} := \{\xi \in \mathcal{E}_B \mid \delta_u(\xi) = 1 \otimes \xi\}$$

to be the vector space of coinvariant elements in \mathcal{E}_B . In this case the right $B \otimes \mathcal{A}(S_\theta^4)$ -module structure on \mathcal{E}_A survives as a right $M \otimes \mathcal{A}_r(S_\theta^4)$ -module structure on \mathcal{E}_M .

Lemma 7.9. *The induced module*

$$\mathcal{E}_M \otimes_{M \otimes \mathcal{A}_r(S_\theta^4)} (B \otimes \mathcal{A}_r(S_\theta^4)) \simeq \mathcal{E}_M \otimes_M A$$

is canonically isomorphic to \mathcal{E}_B as a right $B \otimes \mathcal{A}_r(S_\theta^4)$ -module.

Proof. The strategy is the same as in Lemma 7.4, in that the result follows from showing that the canonical algebra inclusion $\iota : M \hookrightarrow B$ is a faithfully flat braided Hopf-Galois extension. Associated to the coaction δ_u we have the corresponding canonical linear map

$$\chi_u : B \otimes_M B \rightarrow H_F \otimes B, \quad b \otimes b' \rightarrow \delta_u(b)b',$$

which we would like to show is a bijection. Since H_F is cosemisimple as a coalgebra, it is sufficient to check that χ_u is surjective [20]. Moreover, it is known that χ_u is surjective if, whenever h is a generator of H_F , then the element $1 \otimes h$ is in its image [19]. The canonical map here works out on finitely-generated elements of H_F to be

$$\chi_u((1 \otimes h) \otimes (1 \otimes h')) = 1 \otimes hh' \otimes (h')^*,$$

so that in order to find τ_j in the image of χ_u for each $j = 1, \dots, 4$ we can simply take $h = \tau_j$ and $h' = \tau_j^*$. \square

The module \mathcal{E}_M defines a family of Hermitian vector bundles over S_θ^4 parameterised by the algebra $M = \mathcal{A}(M_\theta^u)$. As before, there is a corresponding family of instantons parameterised by this space.

Proposition 7.10. *Let $\iota : M \hookrightarrow B$ be the canonical algebra inclusion. There exists a Grassmann family of instantons ∇_u parameterised by the algebra $\mathcal{A}(M_\theta^u)$, unique up to unitary equivalence, with the property that $\iota_*(\nabla_u) = \nabla_B$.*

Proof. This follows in exactly the same way as the proof of Proposition 7.5. The coinvariant submodule \mathcal{E}_M is finitely-generated and projective, hence defined by a projection P_u . We define the required family ∇_u of instantons by $\nabla_u := P_u \circ (\text{id} \otimes d)$. \square

Proposition 7.11. *Let $u = (u_1, u_2, u_3, u_4)$ and $v = (v_1, v_2, v_3, v_4)$ be quadruples of unitary elements in H_F such that $u_1^* = u_2$, $u_3^* = u_4$, $v_1^* = v_2$, $v_3^* = v_4$. Then the families of instantons ∇_u and ∇_v described by the parameter spaces $\mathcal{A}(M_\theta^u)$ and $\mathcal{A}(M_\theta^v)$ are gauge equivalent.*

Proof. The families ∇_u and ∇_v are defined as in Proposition 7.10, with corresponding parameter spaces $\mathcal{A}(M_\theta^u)$ and $\mathcal{A}(M_\theta^v)$ arising as coinvariant subalgebras for the coactions δ_u and δ_v of H_F on $B = \mathcal{B}(M_\theta) \rtimes H_F$. These parameter spaces each sit inside B via the canonical algebra inclusions $\iota_u : \mathcal{A}(M_\theta^u) \hookrightarrow B$ and $\iota_v : \mathcal{A}(M_\theta^v) \hookrightarrow B$. The result follows from the fact that the coactions δ_u and δ_v are themselves unitarily equivalent, as one may infer from Eq. (6.5), for example. \square

7.3. The space M_θ^u of connections. We would like to compute the algebra $\mathcal{A}(M_\theta^u)$ explicitly. For convenience, we restrict our attention to the following special case. Let $(r_1, r_2) \in \mathbb{Z}^2$ be a pair of integers and take

$$u = (u_1, u_2, u_3, u_4) := (\tau_1^{r_1}, \tau_2^{r_1}, \tau_3^{r_2}, \tau_4^{r_2}).$$

Then for this u , we can arrange the generators M_{ij}^u of the algebra $\mathcal{A}(M_\theta^u)$ into a matrix $M_\theta^u = (M_{ij}^u)$. Explicitly, one finds that

$$M_\theta^u = \begin{pmatrix} m & 0 & g_1 & g_2^* \\ 0 & m & -\bar{\nu}g_2 & \nu g_1^* \\ g_1^* & -\nu g_2^* & n & 0 \\ g_2 & \bar{\nu}g_1 & 0 & n \end{pmatrix}, \quad \nu := \mu^{r_1 - r_2 + 1},$$

where $m := M_{11}^u$, $n := M_{33}^u$, $g_1 := M_{13}^u$, $g_2 := M_{41}^u$ and $\mu := e^{i\pi\theta}$ is the deformation parameter. The relations between these generators depend of course on the choice of integers r_1, r_2 . We compute them as follows.

Proposition 7.12. *The relations between the entries of the matrix M_θ^u in the algebra $\mathcal{A}(M_\theta^u)$ are given by*

$$g_1 g_2 = \nu^2 g_2 g_1, \quad g_1 g_2^* = \bar{\nu}^2 g_2^* g_1, \quad g_1 g_1^* = g_1^* g_1, \quad g_2 g_2^* = g_2^* g_2$$

and m, n central. There is also a quadric relation

$$mn - \bar{\nu} \mu g_1^* g_1 + \nu \bar{\mu} g_2^* g_2 = 1.$$

Proof. Computing the commutation relations is a simple calculation using the relations in the algebra $A = \mathcal{B}(\mathrm{SL}_\theta(2, \mathbb{H})) \rtimes H_F$. For the quadric relation, one computes that $mn = \mathbf{m}_{11}\mathbf{m}_{33} \otimes 1$, $g_1^* g_1 = \mu^{r_2-r_1} \mathbf{m}_{31}\mathbf{m}_{13} \otimes 1$ and $g_2^* g_2 = \mu^{r_1-r_2} \mathbf{m}_{41}\mathbf{m}_{14} \otimes 1$, whence we have that

$$mn - \bar{\nu} \mu g_1^* g_1 + \nu \bar{\mu} g_2^* g_2 = (\mathbf{m}_{11}\mathbf{m}_{33} - \mathbf{m}_{31}\mathbf{m}_{13} + \mathbf{m}_{41}\mathbf{m}_{14}) \otimes 1 = \det(A_\theta) \otimes 1,$$

where A_θ is the θ -deformed version of the matrix in Eq. (4.1). The relation

$$\det(A_\theta) = \mathbf{m}_{11}\mathbf{m}_{33} - \mathbf{m}_{31}\mathbf{m}_{13} + \mathbf{m}_{41}\mathbf{m}_{14}$$

is computed as in [11]. The fact that $\det(A_\theta) = 1$ in $\mathcal{B}(\mathrm{SL}_\theta(2, \mathbb{H}))$ gives the relation as stated. \square

We see that the choice of unitary u affects both the commutation relations in the algebra $\mathcal{A}(M_\theta^u)$ as well as the quadric relation. We emphasise the following two important cases.

Example 7.13. Any choice for which $r_1 = r_2$ recovers the noncommutative parameter space discovered in [11], whose algebra relations have m, n central and $g_1 g_2 = \mu^2 g_2 g_1$, $g_1 g_2^* = \bar{\mu}^2 g_2^* g_1$. The quadric relation is the same as the classical one, $mn - g_1^* g_1 + g_2^* g_2 = 1$.

Example 7.14. For any choice which has $r_2 = r_1 + 1$ we have $\nu = 1$ and hence we obtain a commutative parameter space, *i.e.* the generators m, n, g_1, g_2 and their conjugates all commute. However, the quadric relation in this case is deformed, $mn - \mu g_1^* g_1 + \bar{\mu} g_2^* g_2 = 1$.

It is not difficult to see that, in the classical limit, the algebras $\mathcal{A}(M_\theta^u)$ describe different but gauge equivalent parameterisations of the same space. In the noncommutative case, these parameter spaces are evidently different, some being noncommutative and others classical, but they are nevertheless still all gauge equivalent.

8. INSTANTONS WITH HIGHER TOPOLOGICAL CHARGE

In this section we generalise the previous construction to treat parameter spaces for instantons of higher topological charge. In [3] we gave a deformed version of the ADHM construction which produced noncommutative families of instantons with arbitrary charge. We start with a review of this construction which emphasises how it may be viewed in the context of braided geometry. As we did for the charge one case, we then show how to use gauge theory to recover commutative parameter spaces.

8.1. A noncommutative space of monads. We begin with a description of the space of monads over the classical space \mathbb{C}^4 , which we shall later deform by means of the twisting cocycle F on the torus algebra $H = \mathcal{A}(\mathbb{T}^2)$ that has been used throughout this paper. We adopt the categorical approach used in the charge one case and demand that all of our constructions are H -covariant; the quantisation functor will produce the twisted version.

The algebra $\mathcal{A}(\mathbb{C}^4)$ has a natural \mathbb{Z} -grading given on generators by

$$\deg(z_j) = 1, \quad \deg(z_j^*) = -1, \quad j = 1, \dots, 4.$$

This gives rise to a decomposition into homogeneous subspaces $\mathcal{A}(\mathbb{C}^4) = \bigoplus_{n \in \mathbb{Z}} \mathcal{A}_n$. For each $r \in \mathbb{Z}$ we denote by $\mathcal{A}(\mathbb{C}^4)(r)$ the ‘degree shifted’ algebra, whose degree n component is defined to be \mathcal{A}_{n+r} . Similarly, for each finite dimensional vector space \mathcal{H} the corresponding free right module $\mathcal{H} \otimes \mathcal{A}(\mathbb{C}^4)$ is \mathbb{Z} -graded by the grading on $\mathcal{A}(\mathbb{C}^4)$, and the shift maps on $\mathcal{A}(\mathbb{C}^4)$ induce shift maps on $\mathcal{H} \otimes \mathcal{A}(\mathbb{C}^4)$.

Definition 8.1. Let $k \in \mathbb{Z}$ be a fixed positive integer. A *monad* over the algebra $\mathcal{A}(\mathbb{C}^4)$ is a sequence of free right $\mathcal{A}(\mathbb{C}^4)$ -modules,

$$(8.1) \quad 0 \rightarrow \mathcal{H} \otimes \mathcal{A}(\mathbb{C}^4)(-1) \xrightarrow{\sigma_z} \mathcal{K} \otimes \mathcal{A}(\mathbb{C}^4) \xrightarrow{\tau_z} \mathcal{L} \otimes \mathcal{A}(\mathbb{C}^4)(1) \rightarrow 0,$$

where \mathcal{H} , \mathcal{K} and \mathcal{L} are complex vector spaces of dimensions k , $2k+2$ and k respectively, such that the maps σ_z , τ_z are linear in the generators z_1, \dots, z_4 of $\mathcal{A}(\mathbb{C}^4)$. The first and last terms of the sequence are required to be exact, so that the only non-trivial cohomology is in the middle term.

As in [3], our strategy is to find the space of all possible monads for a fixed choice of positive integer k . We begin by considering the module map σ_z in the complex (8.1). Choosing ordered bases (u_1, \dots, u_k) for the vector space \mathcal{H} and (v_1, \dots, v_{2k+2}) for the vector space \mathcal{K} , we can express σ_z as

$$(8.2) \quad \sigma_z : u_b \otimes Z \mapsto \sum_{a,j} M_{ab}^j \otimes v_a \otimes z_j Z, \quad Z \in \mathcal{A}(\mathbb{C}^4),$$

for $(2k+2) \times k$ matrices $M^j := (M_{ab}^j)$, where $j = 1, \dots, 4$ and $a = 1, \dots, 2k+2$, $b = 1, \dots, k$. Thus, in more compact notation, σ_z may be written

$$(8.3) \quad \sigma_z = \sum_j M^j \otimes z_j.$$

In dual terms, we think of the M_{ab}^j as coordinate functions on the space $M(\mathcal{H}, \mathcal{K})$ of all such maps σ_z , with (commutative) coordinate algebra $\mathcal{A}(M(\mathcal{H}, \mathcal{K}))$ generated by the functions M_{ab}^j for $j = 1, \dots, 4$ and $a = 1, \dots, 2k+2$, $b = 1, \dots, k$. It comes equipped with the homomorphism (8.2) of right $\mathcal{A}(\mathbb{C}^4)$ -modules. In this way, the space $M(\mathcal{H}, \mathcal{K})$ in fact has the structure of an algebraic variety: it is the spectrum of the algebra $\mathcal{A}(M(\mathcal{H}, \mathcal{K}))$.

As mentioned above, we wish to view the construction as taking place in the category ${}^H\mathcal{M}$. The free $\mathcal{A}(\mathbb{C}^4)$ -modules appearing in the complex (8.1) are automatically objects in ${}^H\mathcal{M}$; we need that the maps σ_z , τ_z are morphisms.

Lemma 8.2. *The map $\sigma_z := \sum_\alpha M^\alpha \otimes z_\alpha$ is a morphism in the category ${}^H\mathcal{M}$ if and only if the coordinate functions M_{ab}^j carry the left H -coaction given on generators by*

$$M_{ab}^j \mapsto \tau_j^* \otimes M_{ab}^j,$$

for each $j = 1, \dots, 4$ and $a = 1, \dots, 2k+2$, $b = 1, \dots, k$, making the vector space spanned by the M_{ab}^j into a left H -comodule.

Proof. Upon inspection of Eq. (8.2) we see that σ_z cannot possibly be an intertwiner for the H -coactions on $\mathcal{H} \otimes \mathcal{A}(\mathbb{C}^4)$ and on $\mathcal{K} \otimes \mathcal{A}(\mathbb{C}^4)$ unless we also allow for a coaction of H on the algebra $\mathcal{A}(\mathcal{M}(\mathcal{H}, \mathcal{K}))$ as well. It is clear that, for σ_z to be H -covariant, this coaction needs to be as stated in the lemma. \square

It follows that the algebra $\mathcal{A}(\mathcal{M}(\mathcal{H}, \mathcal{K}))$ is an algebra in the category ${}^H\mathcal{M}$. It possesses a certain universality property which we discuss in Appendix A, reinforcing our assertion that it is the coordinate algebra of the space of all module maps σ_z .

We may carry out the same analysis for the map τ_z . We choose a basis (w_1, \dots, w_k) for the vector space \mathcal{L} and consider the map

$$(8.4) \quad \tau_z : v_a \otimes Z \mapsto \sum_{b,j} N_{ba}^j \otimes w_b \otimes z_j Z.$$

Then the commutative algebra $\mathcal{A}(\mathcal{M}(\mathcal{K}, \mathcal{L}))$ generated by the matrix elements

$$\{N_{ba}^j \mid a = 1, \dots, 2k+2, b = 1, \dots, k, j = 1, \dots, 4\},$$

when equipped with the morphism of right $\mathcal{A}(\mathbb{C}^4)$ -modules

$$\tau_z : \mathcal{K} \otimes \mathcal{A}(\mathbb{C}^4) \rightarrow \mathcal{A}(\mathcal{M}(\mathcal{K}, \mathcal{L})) \otimes \mathcal{L} \otimes \mathcal{A}(\mathbb{C}^4)(1),$$

is the coordinate algebra of the space of all maps $\mathcal{K} \otimes \mathcal{A}(\mathbb{C}^4) \rightarrow \mathcal{L} \otimes \mathcal{A}(\mathbb{C}^4)(1)$. In compact notation, the map τ_z has the form

$$(8.5) \quad \tau_z = \sum_j N^j \otimes z_j$$

upon collecting the generators into the $k \times (2k+2)$ matrices $N^j := (N_{ba}^j)$. For covariance we need the left H -coaction on $\mathcal{A}(\mathcal{M}(\mathcal{K}, \mathcal{L}))$ given by

$$N_{ba}^j \mapsto \tau_j^* \otimes N_{ba}^j$$

for $j = 1, \dots, 4$ and $a = 1, \dots, 2k+2$, $b = 1, \dots, k$, which makes $\mathcal{A}(\mathcal{M}(\mathcal{K}, \mathcal{L}))$ into a left H -comodule algebra, an object in the category ${}^H\mathcal{M}$.

Next we need to address the requirement that (8.1) be a complex, *i.e.* that the composition $\vartheta_z := \tau_z \circ \sigma_z$ is zero. To obtain this in a coordinate algebra framework, we note that the space of all right module maps $\mathcal{H} \otimes \mathcal{A}(\mathbb{C}^4)(-1) \rightarrow \mathcal{L} \otimes \mathcal{A}(\mathbb{C}^4)(1)$ which are quadratic in the generators z_1, \dots, z_4 is encoded by the commutative algebra $\mathcal{A}(\mathcal{M}(\mathcal{H}, \mathcal{L}))$ generated by matrix elements

$$\{T_{cd}^{j,l} \mid c, d = 1, \dots, k, j, l = 1, \dots, 4\},$$

together with the right module map

$$\vartheta_z : \mathcal{H} \otimes \mathcal{A}(\mathbb{C}^4)(-1) \rightarrow \mathcal{A}(\mathcal{M}(\mathcal{H}, \mathcal{L})) \otimes \mathcal{L} \otimes \mathcal{A}(\mathbb{C}^4)(1),$$

$$\vartheta_z : u_b \otimes Z \mapsto \sum_{j,l,d} T_{db}^{j,l} \otimes w_d \otimes z_j z_l Z$$

with respect to our earlier choice of bases. The identification of ϑ_z with the composition $\tau_z \circ \sigma_z$ appears in coordinate form as a ‘coproduct’ or a ‘gluing’ of rectangular matrices [16], *i.e.* as an algebra map

$$\Delta : \mathcal{A}(\mathcal{M}(\mathcal{H}, \mathcal{L})) \rightarrow \mathcal{A}(\mathcal{M}(\mathcal{K}, \mathcal{L})) \otimes \mathcal{A}(\mathcal{M}(\mathcal{H}, \mathcal{K})),$$

$$(8.6) \quad \Delta(T_{cd}^{j,l}) := \sum_b N_{cb}^j \otimes M_{bd}^l, \quad j, l = 1, \dots, 4, \quad c, d = 1, \dots, k.$$

Therefore, requiring that the composition be zero results in the extra relations

$$(8.7) \quad \sum_b (N_{cb}^j \otimes M_{bd}^l + N_{cb}^l \otimes M_{bd}^j) = 0,$$

for all $j, l = 1, \dots, 4, \quad c, d = 1, \dots, k$.

Definition 8.3. We denote by $\mathcal{A}(\tilde{\mathcal{M}}_k)$ the coordinate algebra of the space of all monads (8.1). It is the quotient of the tensor product algebra $\mathcal{A}(\mathcal{M}(\mathcal{K}, \mathcal{L})) \otimes \mathcal{A}(\mathcal{M}(\mathcal{H}, \mathcal{K}))$ by the relations (8.7).

We are now ready to pass to the noncommutative situation. Applying the ‘quantisation functor’ deforms our matrix coordinate algebras according to the following.

Proposition 8.4. *The relations in the algebras $\mathcal{A}(\mathcal{M}(\mathcal{H}, \mathcal{K}))$ and $\mathcal{A}(\mathcal{M}(\mathcal{K}, \mathcal{L}))$ are deformed into*

$$M_{ab}^j M_{cd}^l = \eta_{lj} M_{cd}^l M_{ab}^j, \quad N_{ba}^j N_{dc}^l = \eta_{lj} N_{dc}^l N_{ba}^j$$

for all $j, l = 1, \dots, 4$ and all $a, c = 1, \dots, 2k + 2, \quad b, d = 1, \dots, k$.

Proof. We apply the deformation functor described in §2.2. The products of generators are deformed into $M_{ab}^j \cdot M_{cd}^l = F(\tau_j^*, \tau_l^*) M_{ab}^j M_{cd}^l$ and $N_{ba}^j \cdot N_{dc}^l = F(\tau_j^*, \tau_l^*) N_{ba}^j N_{dc}^l$ respectively, from which the relations in the deformed algebras follow as stated. \square

We denote the resulting H_F -covariant algebras by $\mathcal{B}(\mathcal{M}_\theta(\mathcal{H}, \mathcal{K}))$ and $\mathcal{B}(\mathcal{M}_\theta(\mathcal{K}, \mathcal{L}))$. In turn, the ‘coproduct’ in Eq. (8.1) is deformed into

$$\Delta_F(T_{cd}^{i,j}) = \sum_b N_{cb}^i \otimes M_{bd}^j F^{-1}(\tau_i^*, \tau_j^*),$$

although the extra factor of F^{-1} can be absorbed upon redefining the generators – as we did in Eq. (4.10) – and we shall henceforth assume this has been done, without changing our notation. As was the case for the braided conformal group in §5.2, this Δ_F now extends as a homomorphism to the braided tensor product algebra,

$$\Delta_F : \mathcal{B}(\mathcal{M}_\theta(\mathcal{H}, \mathcal{L})) \rightarrow \mathcal{B}(\mathcal{M}_\theta(\mathcal{K}, \mathcal{L})) \underline{\otimes} \mathcal{B}(\mathcal{M}_\theta(\mathcal{H}, \mathcal{K})),$$

so that imposing that the composition $\tau_z \circ \sigma_z$ is zero now results in the deformed relations

$$(8.8) \quad \sum_r (N_{dr}^j M_{rb}^l + \eta_{jl} N_{br}^l M_{rd}^j) = 0$$

for all $j, l = 1, \dots, 4$ and all $b, d = 1, \dots, k$, just as found in [3].

Definition 8.5. Define $\mathcal{B}(\tilde{\mathcal{M}}_{\theta;k})$ to be the braided tensor product algebra

$$\mathcal{B}(\mathcal{M}_\theta(\mathcal{K}, \mathcal{L})) \underline{\otimes} \mathcal{B}(\mathcal{M}_\theta(\mathcal{H}, \mathcal{K}))$$

modulo the relations (8.8).

We stress that the relations (8.8) are *not* commutation relations between the matrix generators: they are rather a set of quadratic relations in the algebra.

8.2. The noncommutative ADHM construction. The monads described in the previous section are by themselves insufficient for the construction of bundles over S_θ^4 . In the classical case, the cohomology of a monad is naturally a finitely-generated projective right $\mathcal{A}(\mathbb{C}^4)$ -module and hence a bundle over \mathbb{C}^4 . But one needs to ensure that this bundle is the pull-back of some bundle over S^4 , which is achieved by equipping the monad with certain ‘reality structure’; in our deformed setting this is incarnated as a $*$ -structure on the algebra $\mathcal{B}(\tilde{\mathcal{M}}_{\theta;k})$.

For this extra structure, we use the anti-linear map $J : \mathcal{A}(\mathbb{C}_\theta^4) \rightarrow \mathcal{A}(\mathbb{C}_\theta^4)$ defined by

$$(8.9) \quad J(z_1, z_2, z_3, z_4) := (-z_2^*, z_1^*, -z_4^*, z_3^*)$$

and extended as an anti-algebra homomorphism. It is clearly a morphism in the category of left H_F -comodules. For each finite-dimensional complex vector space \mathcal{H} this immediately gives a free left $\mathcal{A}(\mathbb{C}_\theta^4)$ -module $\mathcal{H} \otimes J(\mathcal{A}(\mathbb{C}_\theta^4))$ whose module structure is defined by $Z \triangleright (u \otimes J(W)) := u \otimes J(WZ)$ for each $u \in \mathcal{H}$, $W, Z \in \mathcal{A}(\mathbb{C}_\theta^4)$. Dual to this, we have the free right $\mathcal{A}(\mathbb{C}_\theta^4)$ -module $\mathcal{H}^* \otimes J(\mathcal{A}(\mathbb{C}_\theta^4))^*$, where \mathcal{H}^* is the dual vector space to \mathcal{H} and $J(\mathcal{A}(\mathbb{C}_\theta^4))^* := \text{Hom}_{\mathcal{A}(\mathbb{C}_\theta^4)}(J(\mathcal{A}(\mathbb{C}_\theta^4)), \mathcal{A}(\mathbb{C}_\theta^4))$.

Introducing the conjugate matrix generators M_{ab}^{j*} and N_{cd}^{l*} , we write $(M^{j\dagger})_{ab} = M_{ba}^{j*}$ and $(N^{l\dagger})_{cd} = N_{dc}^{l*}$. All of this gives rise to a ‘dual monad’

$$0 \rightarrow \mathcal{L}^* \otimes J(\mathcal{A}(\mathbb{C}_\theta^4))^*(-1) \xrightarrow{\tau_{J(z)}^*} \mathcal{K}^* \otimes J(\mathcal{A}(\mathbb{C}_\theta^4))^* \xrightarrow{\sigma_{J(z)}^*} \mathcal{H}^* \otimes J(\mathcal{A}(\mathbb{C}_\theta^4))^*(1),$$

where $\tau_{J(z)}^*$ and $\sigma_{J(z)}^*$ are the ‘adjoint’ maps defined by

$$\sigma_{J(z)}^* = \sum_j M^{j\dagger} \otimes J(z_j)^*, \quad \tau_{J(z)}^* = \sum_j N^{j\dagger} \otimes J(z_j)^*.$$

We impose the condition that monads should be self-conjugate with respect to this process, resulting in the $*$ -structure

$$(8.10) \quad N^1 = -M^{2\dagger}, \quad N^2 = M^{1\dagger}, \quad N^3 = -M^{4\dagger}, \quad N^4 = M^{3\dagger}$$

on the algebra $\mathcal{B}(\tilde{\mathcal{M}}_{\theta;k})$. Note that the involution defined in (8.10) is compatible with the H_F -coaction and hence with the algebra relations. Although the algebra relations are slightly different, this construction is otherwise described in more detail in [3].

Definition 8.6. We write $\mathcal{B}(\mathcal{M}_{\theta;k})$ for the quotient of the algebra $\mathcal{B}(\tilde{\mathcal{M}}_{\theta;k})$ by the $*$ -relations in Eq. (8.10). It is the coordinate algebra of the space of self-conjugate monads in the category ${}^{H_F}\mathcal{M}$.

For self-conjugate monads, the important maps are therefore the $(2k+2) \times k$ algebra-valued matrices

$$\begin{aligned} \sigma_z &= M^1 \otimes z_1 + M^2 \otimes z_2 + M^3 \otimes z_3 + M^4 \otimes z_4, \\ \sigma_{J(z)} &= -M^1 \otimes z_2^* + M^2 \otimes z_1^* - M^3 \otimes z_4^* + M^4 \otimes z_3^*, \end{aligned}$$

which obey the monad conditions $\sigma_{J(z)}^* \sigma_z = 0$ and $\sigma_{J(z)}^* \sigma_{J(z)} = \sigma_z^* \sigma_z$. The crucial technical condition that we need for the ADHM construction is the following.

Lemma 8.7. *The entries of the matrix $\rho^2 := \sigma_z^* \sigma_z = \sigma_{J(z)}^* \sigma_{J(z)}$ commute with the entries of the matrix σ_z .*

Proof. One finds that the (μ, ν) entry of ρ^2 and the (a, b) entry of σ_z are respectively

$$(\rho^2)_{\mu\nu} = \sum_{r,j,l} (M^{j\dagger})_{\mu r} M_{r\nu}^l \otimes z_j^* z_l, \quad (\sigma_z)_{ab} = \sum_s M_{ab}^s \otimes z_s.$$

Suppressing the summation, the relations between these elements are computed in the braided tensor product algebra $\mathcal{B}(\mathbf{M}_{\theta;k}) \underline{\otimes} \mathcal{A}(\mathbb{C}_\theta^4)$ as follows:

$$\begin{aligned} ((M^{j\dagger})_{\mu r} M_{r\nu}^l \otimes z_j^* z_l) (M_{ab}^s \otimes z_s) &= (M^{j\dagger})_{\mu r} M_{r\nu}^l M_{ab}^s \otimes z_j^* z_l z_s F^{-2}(\tau_s^*, \tau_j^* \tau_l) \\ &= M_{ab}^s (M^{j\dagger})_{\mu r} M_{r\nu}^l \otimes z_s z_j^* z_l F^{-2}(\tau_s^*, \tau_j^* \tau_l) (\eta_{js} \eta_{sl})^2 \\ &= (M_{ab}^s \otimes z_s) ((M^{j\dagger})_{\mu r} M_{r\nu}^l \otimes z_j^* z_l) \\ &\quad F^{-2}(\tau_s^*, \tau_j^* \tau_l) (\eta_{js} \eta_{sl})^2 F^{-2}(\tau_s, \tau_j \tau_l^*) \\ &= (M_{ab}^s \otimes z_s) ((M^{j\dagger})_{\mu r} M_{r\nu}^l \otimes z_j^* z_l). \end{aligned}$$

In the first and third equalities we have used the definition of the braided tensor product; in the second equality we have used the algebra relations in $\mathcal{B}(\mathbf{M}_{\theta;k})$ and $\mathcal{A}(\mathbb{C}_\theta^4)$. \square

We slightly enlarge the matrix algebra $M_k(\mathbb{C}) \otimes (\mathcal{B}(\mathbf{M}_{\theta;k}) \underline{\otimes} \mathcal{A}(\mathbb{C}_\theta^4))$ by adjoining an inverse element ρ^{-2} for ρ^2 and combine the matrices $\sigma_z, \sigma_{J(z)}$ into the $(2k+2) \times 2k$ matrix

$$(8.11) \quad \mathbf{V} := \begin{pmatrix} \sigma_z & \sigma_{J(z)} \end{pmatrix},$$

which by the definition of ρ^2 obeys

$$\mathbf{V}^* \mathbf{V} = \rho^2 \begin{pmatrix} \mathbb{I}_k & 0 \\ 0 & \mathbb{I}_k \end{pmatrix},$$

where \mathbb{I}_k denotes the $k \times k$ identity matrix. It follows as in [3] that the quantity

$$(8.12) \quad \mathbf{Q} := \mathbf{V} \rho^{-2} \mathbf{V}^* = \sigma_z \rho^{-2} \sigma_z^* + \sigma_{J(z)} \rho^{-2} \sigma_{J(z)}^*$$

is automatically a $(2k+2) \times (2k+2)$ projection, $\mathbf{Q}^2 = \mathbf{Q} = \mathbf{Q}^*$, with entries in the algebra $\mathcal{B}(\mathbf{M}_{\theta;k}) \underline{\otimes} \mathcal{A}_r(S_\theta^4)$. From this we construct the complementary projection $\mathbf{P} := \mathbb{I}_{2k+2} - \mathbf{Q}$, having entries in the same algebra.

At this point we encounter the same technical issue that we did in the charge one case: for \mathbf{P} to define an honest family of vector bundles as in Definition 6.1, we need a projection with entries in an algebra of the form $A \otimes \mathcal{A}_r(S_\theta^4)$ (where A is the parameter space), whereas the projection \mathbf{P} has entries in the *braided* tensor product $\mathcal{B}(\mathbf{M}_{\theta;k}) \underline{\otimes} \mathcal{A}_r(S_\theta^4)$. In the charge one case we had a $\mathcal{B}(\mathrm{SL}_\theta(2, \mathbb{H})) \underline{\otimes} \mathcal{A}_r(S_\theta^4)$ -valued projection, from which we passed to a $(\mathcal{B}(\mathrm{SL}_\theta(2, \mathbb{H})) \rtimes H_F) \otimes \mathcal{A}_r(S_\theta^4)$ -valued projection by making a cobosonisation. Despite the fact that $\mathcal{B}(\mathbf{M}_{\theta;k})$ is only an algebra and not a Hopf algebra, we can nevertheless use the same strategy to obtain a genuine family of vector bundles.

Indeed, we shall convert \mathbf{P} into a projection with entries in the algebra $\mathcal{B}(\mathbf{M}_{\theta;k}) \rtimes H_F$, where the cross product is the one defined by the canonical left action of H_F on $\mathcal{B}(\mathbf{M}_{\theta;k})$ defined in Eq. (2.4) for the general case. This action is given on generators by the formula

$$h \triangleright M_{ab}^j = F^{-2}(\tau_j^*, h) M_{ab}^j, \quad h \triangleright (M_{ab}^j)^* = F^{-2}(\tau_j, h) (M_{ab}^j)^*, \quad h \in H_F,$$

for $j = 1, \dots, 4$ and $a = 1, \dots, 2k+2, b = 1, \dots, k$, and it comes from the left coaction

$$\Delta_L : \mathcal{A}(\mathbb{C}_\theta^4) \rightarrow H_F \otimes \mathcal{A}(\mathbb{C}_\theta^4), \quad z_j \mapsto \tau_j \otimes z_j,$$

extended as a $*$ -algebra map. More generally we shall denote the coaction on an arbitrary element $Z \in \mathcal{A}(\mathbb{C}_\theta^4)$ by $\Delta_L(Z) = Z^{(-1)} \otimes Z^{(0)}$. The key result that we need is the following.

Lemma 8.8. *There is a $*$ -algebra map*

$$\beta : \mathcal{B}(\mathbf{M}_{\theta;k}) \underline{\otimes} \mathcal{A}(\mathbb{C}_\theta^4) \rightarrow (\mathcal{B}(\mathbf{M}_{\theta;k}) \rtimes_{H_F}) \otimes \mathcal{A}(\mathbb{C}_\theta^4)$$

defined by $\beta(M \otimes Z) = M \otimes Z^{(-1)} \otimes Z^{(0)}$, for each $M \in \mathcal{B}(\mathbf{M}_{\theta;k})$ and $Z \in \mathcal{A}(\mathbb{C}_\theta^4)$.

Proof. We simply check that on generators we have

$$\begin{aligned} \beta(M_{ab}^j \otimes z_l) \beta(M_{cd}^r \otimes z_s) &= (M_{ab}^j \otimes \tau_l \otimes z_l) (M_{cd}^r \otimes \tau_s \otimes z_s) \\ &= M_{ab}^j M_{cd}^r \otimes \tau_l \tau_s \otimes z_l z_s F^{-2}(\tau_r^*, \tau_l) \\ &= \beta(M_{ab}^j M_{cd}^r \otimes z_l z_s) F^{-2}(\tau_r^*, \tau_l) \\ &= \beta((M_{ab}^j \otimes z_l)(M_{cd}^r \otimes z_s)) \end{aligned}$$

for all $j, l, r, s = 1, \dots, 4$, showing that β is an algebra map. Moreover,

$$\begin{aligned} (\beta(M_{ab}^j \otimes z_l))^* &= (M_{ab}^j \otimes \tau_l \otimes z_l)^* = M_{ab}^{j*} \otimes \tau_l^* \otimes z_l^* F^{-2}(\tau_j^*, \tau_l) \\ &= \beta(M_{ab}^{j*} \otimes z_l^*) F^{-2}(\tau_j^*, \tau_l) = \beta((M_{ab}^j \otimes z_l)^*), \end{aligned}$$

so that β respects the $*$ -structure as well. \square

Immediately we apply β to the projection \mathbf{P} and, since it is an algebra map, we obtain a projection $\tilde{\mathbf{P}}$ with entries in the algebra $(\mathcal{B}(\mathbf{M}_{\theta;k}) \rtimes_{H_F}) \otimes \mathcal{A}_r(S_\theta^4)$. In the same way as it is shown in [3], the projection $\tilde{\mathbf{P}}$ defines a family of rank two Hermitian vector bundles over S_θ^4 , together with the family $\tilde{\nabla} := \mathbf{P} \circ (\text{id} \otimes d)$ of Grassmann connections whose curvature is anti-self-dual. The Chern classes of $\tilde{\mathbf{P}}$ are computed to be $\text{ch}_1(\tilde{\mathbf{P}}) = 0$, $\text{ch}_2(\tilde{\mathbf{P}}) = -k$, whence we get a family of charge k instantons parameterised by the algebra $\mathcal{B}(\mathbf{M}_{\theta;k}) \rtimes_{H_F}$.

8.3. Removing the H_F gauge parameters. We may now apply the strategy of §7.2 in order to remove the gauge freedom corresponding to the Hopf algebra H_F from the family $\tilde{\nabla}$. To do this, we have to: choose a coaction of H_F on the parameter space $\mathcal{B}(\mathbf{M}_{\theta;k}) \rtimes_{H_F}$; check that this coaction corresponds to gauge freedom; find the quotient space and verify that it does indeed parameterise a family of instantons.

Proposition 8.9. *Let $u := (u_1, u_2, u_3, u_4)$ be unitary elements of H_F such that $u_1^* = u_2$, $u_3^* = u_4$. Then there is a braided left coaction $\delta_u : \mathcal{B}(\mathbf{M}_{\theta;k}) \rtimes_{H_F} \rightarrow H_F \underline{\otimes} (\mathcal{B}(\mathbf{M}_{\theta;k}) \rtimes_{H_F})$ defined by*

$$\delta_u(M_{ab}^j \otimes h) = u_j^* h \otimes M_{ab}^j \otimes h,$$

for group-like elements $h \in H_F$ and extended as a braided $*$ -algebra map. Moreover, the resulting projection $\delta_u(\tilde{\mathbf{P}})$ is unitarily equivalent to the projection $1 \otimes \tilde{\mathbf{P}}$ in the algebra $M_4(H_F \underline{\otimes} (\mathcal{B}(\mathbf{M}_{\theta;k}) \rtimes_{H_F}) \otimes \mathcal{A}(S_\theta^4))$.

Proof. One verifies the conditions required for δ_u to define a braided H_F -comodule algebra, in the same way as was done in Proposition 7.6. The unitary equivalence is checked in the same way as in the proof of Proposition 7.7. \square

Proposition 8.10. *The subalgebra $\mathcal{A}(\mathbf{M}_{\theta;k}^u)$ of coinvariants in $\mathcal{B}(\mathbf{M}_{\theta;k}) \rtimes_{H_F}$ for the coaction δ_u is generated by elements of the form $M_{ab}^j \otimes u_j$.*

Proof. Clearly one has for all group-like elements $h \in H_F$ that

$$\delta_u(M_{ab}^j \otimes h) = u_j^* h \otimes M_{ab}^j \otimes h,$$

whence for coinvariants we need to take $h = u_j$. \square

We can explicitly compute the relations between generators of the algebra $\mathcal{A}(\mathbf{M}_{\theta;k}^u)$ using the algebra relations of $\mathcal{B}(\mathbf{M}_{\theta;k}) \rtimes H_F$, obtaining

$$(M_{ab}^j \otimes u_j)(M_{cd}^l \otimes u_l) = (M_{cd}^l \otimes u_l)(M_{ab}^j \otimes u_j)F^{-2}(u_l, \tau_j^*)F^{-2}(\tau_l, \tau_j)F^{-2}(\tau_l^*, u_j).$$

In particular, we can take $(r_1, r_2) \in \mathbb{Z}^2$ to be a pair of integers and set

$$u = (u_1, u_2, u_3, u_4) := (\tau_1^{m_1}, \tau_2^{m_2}, \tau_3^{m_3}, \tau_4^{m_4})$$

with $(m_1, m_2, m_3, m_4) := (r_1, r_1, r_2, r_2)$, as we did in the charge one case. For such u , the commutation relations in $\mathcal{A}(\mathbf{M}_{\theta;k}^u)$ reduce to

$$(M_{ab}^j \otimes u_j)(M_{cd}^l \otimes u_l) = \eta_{jl}^{m_l+m_j-1}(M_{cd}^l \otimes u_l)(M_{ab}^j \otimes u_j).$$

Let us check that there is a choice of integers r_1, r_2 for which the parameter space $\mathcal{A}(\mathbf{M}_{\theta;k}^u)$ is commutative. It is easy to see from Eq. (3.8) that, whenever both $j, l \in \{1, 2\}$ or $j, l \in \{3, 4\}$, the deformation parameter η_{jl} is automatically equal to 1 and so these generators always commute. Without loss of generality we consider the non-trivial case $j \in \{1, 2\}$ and $l \in \{3, 4\}$, where the corresponding generators fail to commute by a factor of $\eta_{jl}^{m_l+m_j-1}$. By assumption we have that $m_l = r_1$ and $m_j = r_2$, so it follows that any choice of r_1, r_2 for which $r_1 + r_2 = 1$ makes the resulting algebra $\mathcal{A}(\mathbf{M}_{\theta;k}^u)$ commutative.

Of course, for these parameter spaces there is a great deal of gauge freedom left to be removed. As shown in [3], the ADHM construction does not depend on the choice of bases for the vector spaces \mathcal{H}, \mathcal{L} in the monad (8.1), whereas making a unitary change of basis of \mathcal{K} which respects the self-conjugacy property of the monad (*i.e.* acting with an element of the unitary group $\mathrm{Sp}(\mathcal{K})$) results in a projection which is unitarily equivalent to $\tilde{\mathbf{P}}$. Removing the extra gauge parameters corresponding to these degrees of freedom ought to be straightforward, since the computation is entirely classical, although we postpone an explicit computation to future work.

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APPENDIX A. QUANTUM FAMILIES OF MAPS

In this appendix we briefly review the notion of representability of functors and the corresponding notion of universal objects. These are of paramount importance in the present article, since they tie together the various notions of universality that we use.

Let \mathfrak{C} be a (locally small) category; for each pair of objects A, B of \mathfrak{C} , we write $\mathrm{Mor}(A, B)$ for the set of morphisms from A to B . Let $\mathcal{F} : \mathfrak{C} \rightarrow \mathbf{Set}$ be a covariant functor from \mathfrak{C} to the category \mathbf{Set} of sets.

Definition A.1. A *representation* of the functor \mathcal{F} is a pair (M, Φ) , where M is an object of \mathfrak{C} and $\Phi : \text{Mor}(M, -) \rightarrow \mathcal{F}$ is an isomorphism of functors (*i.e.* a natural transformation whose component morphisms are all isomorphisms). If such a representation (M, Φ) exists, then the functor \mathcal{F} is said to be *representable*.

From Yoneda's lemma one knows that natural transformations from $\text{Mor}(M, -)$ to \mathcal{F} are in bijective correspondence with elements of $\mathcal{F}(M)$ [12]. Indeed, given a natural transformation $\Phi : \text{Mor}(M, -) \rightarrow \mathcal{F}$, there is a corresponding element $\sigma \in \mathcal{F}(M)$ defined by $\sigma := \Phi_M(\text{id}_M)$. Conversely, given $\sigma \in \mathcal{F}(M)$, we can define a natural transformation $\Phi : \text{Mor}(M, -) \rightarrow \mathcal{F}$ by

$$\Phi_X(\delta) := (\mathcal{F} \circ \delta)(\sigma), \quad \text{for } \delta \in \text{Mor}(M, X).$$

This leads to the following definition.

Definition A.2. A *universal object* for the functor \mathcal{F} is a pair (M, σ) , where M is an object of \mathfrak{C} and σ is an element of the set $\mathcal{F}(M)$ with the property that for every pair (Y, ν) with Y an object of \mathfrak{C} and ν an element of $\mathcal{F}(Y)$, there is a unique morphism $\Lambda \in \text{Mor}(M, Y)$ such that $(\mathcal{F} \circ \Lambda)(\sigma) = \nu$.

From the above argument it follows that representations of \mathcal{F} are in one-to-one correspondence with universal objects for \mathcal{F} . Of course, it is not necessarily the case that a functor is representable, but if so, the corresponding universal object is unique up to a unique isomorphism. This abstract categorical set-up is extremely useful when applied to the following examples.

Example A.3. First we recall the instanton moduli functor $\mathcal{F} : \text{Alg} \rightarrow \text{Set}$ defined in Remark 6.8, which assigns to each unital $*$ -algebra A the set $\mathcal{F}(A)$ of equivalence classes of families of instantons parameterised by A . To be more precise, we can define a functor \mathcal{F}_k by considering only families of instantons with a fixed topological charge k . A (fine) moduli space of charge k instantons is a universal object representing the functor \mathcal{F}_k .

Clearly, this set-up is usually far too naive for such a moduli space to exist even in the classical case, but this example is sufficient to illustrate why one should allow for the possibility of noncommutative moduli spaces. The moduli space is necessarily an object in the source of the functor \mathcal{F}_k so that, when allowing noncommutative parameter spaces, one also needs to allow for the possibility of noncommutative moduli spaces.

Example A.4. Let \mathfrak{C} be the category whose objects are unital C^* -algebras. For any two objects A and B the set of morphisms $\text{Mor}(A, B)$ is the set of all non-degenerate $*$ -homomorphisms from A to B . Fix a pair of objects A, B of \mathfrak{C} and define a functor $\mathcal{F} : \mathfrak{C} \rightarrow \text{Set}$ by setting

$$\mathcal{F}(C) := \text{Mor}(B, C \otimes A),$$

i.e. we assign to each C^* -algebra C the set of all morphisms $\delta_C : B \rightarrow C \otimes A$. We say [22] that δ_C is a *quantum family of maps* labeled by C .

In this case a universal object for \mathcal{F} is a C^* -algebra M equipped with a morphism $\delta \in \text{Mor}(B, M \otimes A)$ such that, for any C^* -algebra C and any quantum family of maps δ_C

labeled by C , there exists a unique morphism $\Lambda \in \text{Mor}(M, C)$ and the diagram

$$\begin{array}{ccc} B & \xrightarrow{\delta} & M \otimes A \\ \downarrow \text{id} & & \downarrow \Lambda \otimes \text{id} \\ B & \xrightarrow{\delta_C} & C \otimes A \end{array}$$

is commutative. When A, B, C are commutative C^* -algebras, there exist compact Hausdorff topological spaces $\Omega_A, \Omega_B, \Omega_C$ such that $A = C(\Omega_A)$ and so on. A morphism $\delta_C \in \text{Mor}(B, C \otimes A)$ corresponds to a continuous map $\Omega_A \times \Omega_C \rightarrow \Omega_B$, *i.e.* a continuous family of maps from Ω_A to Ω_B parameterised by the space Ω_C . As explained in [21], the universal object corresponds to the space of *all* continuous maps from Ω_A to Ω_B . The situation where the C^* -algebras are noncommutative is a natural generalisation of this.

As we have seen in the present article, one does not always need to consider C^* -completions and can usually work perfectly well at the level of $*$ -algebras (this is also the usual setting for the algebraic theory of quantum groups [15]). As usual, we think of a noncommutative $*$ -algebra A as the algebras of coordinate functions on some underlying ‘noncommutative space’ Ω_A , with the $*$ -structure interpreted as viewing Ω_A as a ‘real form’ of some complex affine space (see [4] for further discussion in this direction). Of course, one can add more structure if one wishes, such as requiring A to be Noetherian if one wants something resembling a ‘ $*$ -algebraic variety’, although we shall be deliberately vague about this point.

Example A.5. Let \mathfrak{C} be the category of unital $*$ -algebras, with morphisms given by non-degenerate $*$ -homomorphisms. The general principle of the previous example still applies, now in the setting of $*$ -algebraic geometry. Given a pair A, B of objects in the category, an element $\delta_C \in \text{Mor}(B, C \otimes A)$ is a quantum family of maps from Ω_A to Ω_B parameterised by the noncommutative ‘space’ Ω_C .

Example A.6. In the situation of the previous example, we set $B = A$ and define a functor $\mathcal{F} : \mathfrak{C} \rightarrow \mathbf{Set}$ by assigning to each object C the set of all non-degenerate $*$ -algebra maps $\delta_C : A \rightarrow C \otimes A$. One may show [21] that the universal object (M, δ) is automatically a bialgebra, whose coproduct and counit we denote Δ_M, ϵ_M , and that it obeys the additional properties

$$(\text{id} \otimes \delta) \circ \delta = (\Delta_M \otimes \text{id}) \circ \delta, \quad (\epsilon_M \otimes \text{id}) \circ \delta = \text{id},$$

i.e. δ makes A into a left M -comodule algebra. We say that a pair (C, δ_C) obeying these properties is a *transformation bialgebra* for the algebra A . The universal object is called the *universal transformation bialgebra* [11]. In the classical case, it just corresponds to the semigroup of all algebraic maps from the commutative space Ω_A to itself.

Example A.7. Let H be a coquasitriangular Hopf $*$ -algebra and take \mathfrak{C} to be the category of left H -comodule algebras which, as discussed in §2.1, is a braided monoidal category. Once again we fix an algebra A in the category, but we take now the functor $\mathcal{F} : \mathfrak{C} \rightarrow \mathbf{Set}$ to be the one which assigns to each object C of \mathfrak{C} the set of all *braided* morphisms $\delta_C : A \rightarrow C \underline{\otimes} A$, where $\underline{\otimes}$ is the tensor product induced by the braiding. The universal object is the *universal braided transformation bialgebra* for A .

This is the strategy we adopted in §4, where we took $H = \mathcal{A}(\mathbb{T}^2)$ and $A = \mathcal{A}(\mathbb{C}^4)$. With the additional requirement that the bialgebra must respect the quaternionic structure of $\mathbb{H}^2 \simeq \mathbb{C}^4$ (hence inducing the $*$ -structure in Eq. (4.1) as in [11]), we found that the universal transformation bialgebra in the category is the matrix bialgebra $\mathcal{A}(M(2, \mathbb{H}))$. The fact that the quantisation functor is an isomorphism of braided monoidal categories means that it preserves the universality property, so that now viewing \mathcal{F} as a functor from the category of left H_F -comodule algebras to the category of sets, the universal transformation bialgebra for $\mathcal{A}(\mathbb{C}_\theta^4)$ is the braided matrix bialgebra $\mathcal{B}(M_\theta(2, \mathbb{H}))$ of §4.2.

Our final example concerns the construction of parameter spaces of module maps as universal objects. It is more general than the previous examples, which considered algebra maps, but it still uses a universality property to define the ‘space of all maps’. The example illustrates that if one changes the source category of a functor then the problem of representability can alter dramatically. We stress once again that in looking for moduli spaces of instantons, our philosophy is to look not for a set of objects but rather for a space which parameterises those objects, that is to say we ask for some geometric structure. In categorical terms, this means defining a functor from the category of algebras to the category of sets and then looking for the moduli space as a universal object, which is by definition an object in the source category and so necessarily an algebra.

Example A.8. In §8.1, we considered right module maps

$$\sigma_z : \mathcal{H} \otimes \mathcal{A}(\mathbb{C}^4)(-1) \rightarrow \mathcal{K} \otimes \mathcal{A}(\mathbb{C}^4)$$

which are linear in the generators z_1, \dots, z_4 of $\mathcal{A}(\mathbb{C}^4)$ and then looked for the space of all such maps. To view this in a categorical setting, we take \mathfrak{C} *a priori* to be the category of unital algebras ($*$ -structures are not required at this stage) and consider the functor $\mathcal{F} : \mathfrak{C} \rightarrow \mathbf{Set}$ which assigns to each algebra C the set of all right $\mathcal{A}(\mathbb{C}^4)$ -module maps

$$\delta_C : \mathcal{H} \otimes \mathcal{A}(\mathbb{C}^4)(-1) \rightarrow C \otimes \mathcal{K} \otimes \mathcal{A}(\mathbb{C}^4)$$

which are linear in the generators z_1, \dots, z_4 of $\mathcal{A}(\mathbb{C}^4)$. We would like to find the space of all maps σ_z in terms of a universal object for this functor (*i.e.* by proving that it is representable). Following the approach taken in §8.1 and in the above examples, we try and prove representability in this case by explicitly constructing the universal algebra. It is straightforward to see that the universal algebra, if it exists, must be generated by the elements M_{ab}^α which define the map

$$(A.1) \quad \sigma_z : u_b \otimes Z \mapsto \sum_{a,\alpha} M_{ab}^\alpha \otimes v_a \otimes z_\alpha Z, \quad Z \in \mathcal{A}(\mathbb{C}^4),$$

where $j = 1, \dots, 4$ and $a = 1, \dots, 2k + 2$, $b = 1, \dots, k$. In our approach, we need to find an algebra structure on this set of functions M_{ab}^j . However, the construction fails at this point: the objects $\mathcal{H} \otimes \mathcal{A}(\mathbb{C}^4)$ and $\mathcal{K} \otimes \mathcal{A}(\mathbb{C}^4)$ are only $\mathcal{A}(\mathbb{C}^4)$ -modules and do not themselves have an algebra structure, so there is nothing to determine an algebra structure on the matrix elements M_{ab}^j and one has to make a choice.

One could alternatively consider looking for the set of all module maps simply as a vector space, rather than looking for its coordinate algebra. However, this is not very natural as it does not imply any geometric structure; also there does not seem to exist a corresponding notion of universality.

One way to proceed is to look for the space of all such maps σ_z as a classical object, just as we did in §8.1. This means taking the source category \mathfrak{C} of the functor \mathcal{F} to be the category of *commutative* unital algebras: it is perfectly natural to assume in this way that the algebra $\mathcal{A}(\mathcal{M}(\mathcal{H}, \mathcal{K}))$ generated by the coordinate functions M_{ab}^j is commutative, hence giving the space $\mathcal{M}(\mathcal{H}, \mathcal{K})$ the structure of a classical algebraic variety. By restricting the functor in this way, it becomes representable with $\mathcal{A}(\mathcal{M}(\mathcal{H}, \mathcal{K}))$ as the universal object.

Now that we have constructed $\mathcal{A}(\mathcal{M}(\mathcal{H}, \mathcal{K}))$ as a suitable parameter space of maps, we proceed just as in §8.1 to show that, in fact, this algebra is an object in the category of left H -comodules. Viewed in this way, the noncommutative parameter spaces that we construct are just the canonical deformations of the corresponding classical objects and so, in this sense, they are the most natural objects to work with.

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